ESSLLI Tutorial: Nonmonotonic Logic

Default Logic

Mathieu Beirlaen[1] Christian Straßer[1,2]

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[1]Institute for Philosophy II, Ruhr-University Bochum in

[2]Center for Logic and Philosophy of Science, Ghent University

Aims of this session

- learn about the basic ideas behind Reiter's Default Logic
- learn about some of its shortcomings
- … and variants inspired by them

Default Logic - Basic Concepts

Default Logic - Basic Concepts

Warming up

Some References to Classical Articles

- A logic for default reasoning. Artificial Intelligence, 1–2(13). Reiter (1980)
- A logical framework for default reasoning. Artificial intelligence, 36(1), 27–47. Poole (1988)
- The effect of knowledge on belief: conditioning, specificity and the lottery paradox in default reasoning. Artificial Intelligence, 49(1-3), 281–307. Poole (1991)
- Considerations on default logic: an alternative approach. Computational intelligence, 4(1), 1–16. Łukaszewicz (1988)
- Bridges from classical to nonmonotonic logic, chapter 4. Makinson (2005)

Short Reminder: 1st order logic

Logical symbols

- quantifiers $\forall, \exists$
- logical connectives $\wedge, \vee, \supset, \neg$
- brackets
- variables

non-logical symbols

- predicate / relation symbols with specific arity
- function symbols with specific arity
- constants (0-ary functions)

Short Reminder: 1st order logic, special terminology

- terms: variables, $f(t_1, \ldots, t_n)$ where t_i are terms
- atomic formula: $P(t_1, \ldots, t_n)$
- formulas: $\langle \forall, \exists, \wedge, \vee, \supset, \neg \rangle$ closure of atomic formulas
- free / bound variables
- sentence: formula without free variables
- instance of a formula φ: substitution of some free variables for terms
- ground term: term without variables
- ground instance: instance that is a sentence (obtained by substituting all free variables by ground terms)

Example

$$\mathrm{bird}(\text{Tweety}) \supset \mathrm{flies}(\text{Tweety})$$

is a ground instance of

$$\mathrm{bird}(x) \supset \mathrm{flies}(x)$$

Default Logic - Basic Concepts

Defaults and Default Theories

What's a default conditional

prerequisite

$$\frac{\alpha(\mathsf{x}) \quad : \quad \beta_1(\mathsf{x}), \ldots, \beta_n(\mathsf{x})}{\gamma(\mathsf{x})}$$

where $\mathsf{x} = x_1, \ldots, x_m$, and $\alpha(\mathsf{x}), \beta_1(\mathsf{x}), \ldots, \beta_n(\mathsf{x}), \gamma(\mathsf{x})$ are formulas whose free variables are among $x_1, \ldots, x_m$.

What's a default conditional

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justification

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$$\frac{\alpha(\mathsf{x}) \quad : \quad \beta_1(\mathsf{x}), \ldots, \beta_n(\mathsf{x})}{\gamma(\mathsf{x})}$$

conclusion

where $\mathsf{x} = x_1, \ldots, x_m$, and $\alpha(\mathsf{x}), \beta_1(\mathsf{x}), \ldots, \beta_n(\mathsf{x}), \gamma(\mathsf{x})$ are formulas whose free variables are among $x_1, \ldots, x_m$.

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Application of a default

The default is applied in order to derive the $\mathbf{c}$-ground instance of γ in case

- trigger: $\alpha(\mathbf{c})$ belongs to our set of beliefs
- justification: the set of our beliefs is consistent with each $\beta_i(\mathbf{c})$

Default Theory

$$\langle \Delta, \Phi \rangle$$

set of defaults

Default Theory

set of 'facts'

$$\langle \Delta, \Phi \rangle$$

set of defaults

Default Theory

set of 'facts'

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set of defaults

simple example

- $\Delta = \left\{ \dfrac{\mathrm{bird}(x) \quad : \quad \mathrm{flies}(x)}{\mathrm{flies}(x)} \right\}$
- $\Phi = \{\mathrm{bird}(\mathsf{Tweety}), \mathrm{cat}(\mathsf{Sylvester})\}$

Types of defaults

Normal defaults

$$\frac{\alpha(\mathsf{x}) \quad : \quad \gamma(\mathsf{x})}{\gamma(\mathsf{x})}$$

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$$\frac{\alpha(\mathsf{x}) \quad : \quad \gamma(\mathsf{x})}{\gamma(\mathsf{x})}$$

Semi-Normal defaults

$$\frac{\alpha(\mathsf{x}) \quad : \quad \beta(\mathsf{x})}{\gamma(\mathsf{x})}$$

where $\beta(\mathsf{x}) \vdash \gamma(\mathsf{x})$. E.g.,

$$\frac{\alpha(\mathsf{x}) \quad : \quad \gamma(\mathsf{x}) \land \beta(\mathsf{x})}{\gamma(\mathsf{x})}$$

How to Reason with Default Theories

How to Reason with Default Theories

Determining Extensions

Idea: Apply iteratively modus ponens to defaults. This way build step-wise an extension (sets of beliefs that are obtained in this way)

Here's how it goes for a default theory $\langle \Delta, \Phi \rangle$

- guess the extension Ξ

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- (†) take an $\mathbf{c}$-ground instance of an (unused) default

$$\frac{\alpha(\mathbf{x}) \quad : \quad \beta_1(\mathbf{x}), \ldots, \beta_n(\mathbf{x})}{\gamma(\mathbf{x})} \in \Delta$$

  and check whether:

Here's how it goes for a default theory $\langle \Delta, \Phi \rangle$

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  and check whether:

  1. trigger?: $\Xi^\star \vdash \alpha(\mathsf{c})$

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  and check whether:

  1. trigger?: $\Xi^\star \vdash \alpha(\mathbf{c})$
  2. conflicted?: each $\beta_i(\mathbf{c})$ $(1 \leq i \leq n)$ is consistent with Ξ

here the guess is used!

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- if yes: update beliefs: $\Xi^\star := \Xi^\star \cup \{\gamma(\mathbf{c})\}$

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- if yes: update beliefs: $\Xi^\star := \Xi^\star \cup \{\gamma(\mathbf{c})\}$
- if no:
  - try another triggered (unused) default in Δ (goto (†))

here the guess is used!

Here's how it goes for a default theory $\langle \Delta, \Phi \rangle$

- guess the extension Ξ
- init beliefs: $\Xi^\star = \Phi$
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  - if there isn't: terminate.

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  - if $\Xi = \mathrm{Cn}(\Xi^\star)$: extension found .

run may be unsuccessful!

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Problem *Quasi*-Induction – End-regulated procedure: We have to guess and use our guess when adding new defaults.

Example: Tweety

Let $T = \langle \Delta, \Phi \rangle$ where

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Building up the extensions:

- guess: $\Xi = \mathrm{Cn}(\{\mathrm{flies}(\text{Tweety})\} \cup \Phi\})$

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- note that the Sylvester-instance of our default is not applicable to Φ since $\Phi \nvdash \mathrm{bird}(\text{Sylvester})$
- however, we have $\mathrm{bird}(\text{Tweety})$ and $\mathrm{flies}(\text{Tweety})$ is consistent with Ξ.

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- guess: $\Xi = \mathrm{Cn}(\{\mathrm{flies}(\text{Tweety})\} \cup \Phi\})$
- our initial knowledge is Φ
- note that the Sylvester-instance of our default is not applicable to Φ since $\Phi \nvdash \mathrm{bird}(\text{Sylvester})$
- however, we have $\mathrm{bird}(\text{Tweety})$ and $\mathrm{flies}(\text{Tweety})$ is consistent with Ξ.
- fixed point reached
- the only extension is Ξ.

Compact representation

Given a default theory $\langle \Delta, \Phi \rangle$, Ξ is an extension iff $\Xi = \mathrm{Cn}(\bigcup_{i=1}^{\infty} \Xi_i)$ where

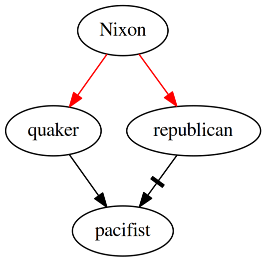
1. $\Xi_0 = \Phi$
2. $\Xi_{i+1} = \Xi_i \cup \left\{ \gamma(\mathsf{c}) \mid \frac{\alpha(x) \ : \ \beta_1(x), \ldots \beta_n(x)}{\gamma(x)} \in \Delta, \Xi_i \vdash \alpha(\mathsf{c}), \neg\beta_1(\mathsf{c}), \ldots, \neg\beta_n(\mathsf{c}) \notin \Xi \right\}$

How to Reason with Default Theories

Extensions and their existence

Let's see: we could define that A is a consequence of the default theory $\langle \Delta, \Phi \rangle$ iff A is in "its extension".

The Nixon Diamond



Let $T = \langle \Delta, \Phi \rangle$ where

- $\Delta = \left\{ \frac{\text{quaker}(x) \; : \; \text{pacifist}(x)}{\text{pacifist}(x)}, \frac{\text{republican}(x) \; : \; \neg\text{pacifist}(x)}{\neg\text{pacifist}(x)} \right\}$
- $\Phi = \{\text{quaker}(\text{Nixon}), \text{republican}(\text{Nixon})\}.$

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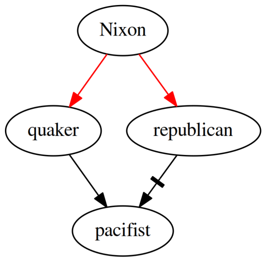

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- $\Phi = \{\text{quaker}(\text{Nixon}), \text{republican}(\text{Nixon})\}$.

There are two extensions:

1. one that contains $\text{pacifist}(\text{Nixon})$,
2. and one that contains $\neg\text{pacifist}(\text{Nixon})$.

… and, do they always exist?

Another example

Let $\langle \Delta, \Phi \rangle$ be a default theory where

- $\Delta = \left\{ \frac{\alpha(x) \; : \; \beta(x)}{\gamma(x)}, \frac{\gamma(x) \; : \; \neg\beta(x)}{\neg\beta(x)} \right\}$
- $\Phi = \{\alpha(\mathsf{c})\}$

Guess: $\mathrm{Cn}(\{\alpha(\mathsf{c}), \gamma(\mathsf{c})\})$

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- first run:

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Guess: $\mathrm{Cn}(\{\alpha(\mathsf{c}), \gamma(\mathsf{c})\})$

- first run:
- $\Phi^\star = \Phi$

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Let $\langle \Delta, \Phi \rangle$ be a default theory where

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- $\Phi = \{\alpha(\mathsf{c})\}$

Guess: $\mathrm{Cn}(\{\alpha(\mathsf{c}), \gamma(\mathsf{c})\})$

- first run:
- $\Phi^\star = \Phi$
- $\frac{\alpha(x) \ : \ \beta(x)}{\gamma(x)}$ is triggered and justified: apply

Another example

Let $\langle \Delta, \Phi \rangle$ be a default theory where

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- $\Phi = \{\alpha(\mathsf{c})\}$

Guess: $\mathrm{Cn}(\{\alpha(\mathsf{c}), \gamma(\mathsf{c})\})$

- first run:
- $\Phi^\star = \Phi$
- $\frac{\alpha(x) \;:\; \beta(x)}{\gamma(x)}$ is triggered and justified: apply
- $\Phi^\star = \Phi \cup \{\gamma(\mathsf{c})\}$

Another example

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Guess: $\mathrm{Cn}(\{\alpha(c), \gamma(c)\})$

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- second run:

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- $\Phi^\star = \Phi$
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- second run:
- $\frac{\gamma(x) \ : \ \neg\beta(x)}{\neg\beta(x)}$ is triggered and justified

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- second run:
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Guess: $\mathrm{Cn}(\{\alpha(c), \gamma(c)\})$

- first run:
- $\Phi^\star = \Phi$
- $\frac{\alpha(x) \;:\; \beta(x)}{\gamma(x)}$ is triggered and justified: apply
- $\Phi^\star = \Phi \cup \{\gamma(c)\}$

- second run:
- $\frac{\gamma(x) \;:\; \neg\beta(x)}{\neg\beta(x)}$ is triggered and justified
- $\Phi^\star = \Phi \cup \{\gamma(c), \neg\beta(c)\}$
- our guess is wrong (similar problems with other guesses)

Reflections about Extensions

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    - good reasons approach: if *A* is in an extension then there are good reasons to suppose *A* (see Nixon)
- are some extensions preferable to others?
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- how to build extensions (naturally) in order to explicate actual default reasoning?

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    - good reasons approach: if A is in an extension then there are good reasons to suppose A (see Nixon)
- are some extensions preferable to others?
- should some extensions be filtered out?
- how to build extensions (naturally) in order to explicate actual default reasoning?
- should floating conclusions be accepted?

How to define the consequences of a default theory?

Two approaches:

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Two approaches:

Skeptical approach

$$\langle \Delta, \Phi \rangle \vdash_{skp} A \text{ iff } A \in \bigcap \mathrm{Extensions}(\langle \Delta, \Phi \rangle)$$

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Two approaches:

Skeptical approach

$$\langle \Delta, \Phi \rangle \vdash_{skp} A \text{ iff } A \in \bigcap \mathrm{Extensions}(\langle \Delta, \Phi \rangle)$$

Credulous approach

$$\langle \Delta, \Phi \rangle \vdash_{crd} A \text{ iff } A \in \bigcup \mathrm{Extensions}(\langle \Delta, \Phi \rangle)$$

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Two approaches:

Skeptical approach

$$\langle \Delta, \Phi \rangle \vdash_{skp} A \text{ iff } A \in \bigcap \mathrm{Extensions}(\langle \Delta, \Phi \rangle)$$

Credulous approach

$$\langle \Delta, \Phi \rangle \vdash_{crd} A \text{ iff } A \in \bigcup \mathrm{Extensions}(\langle \Delta, \Phi \rangle)$$

Question:

When is which approach useful?

Alternative Approaches to Reiter’s

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Makinson's approach

- first: order ground instances of defaults in Δ: $d_1, d_2, \ldots$
- init beliefs: $\Xi_0 = \Phi$ and init used defaults $\Delta_0 = \emptyset$
- in the n+1th step proceed as follows:
    - if there is a $\mathbf{c}$-ground instance of default $\frac{\alpha(\mathbf{c}) \ : \ \beta_1(\mathbf{c}),\ldots,\beta_m(\mathbf{c})}{\gamma(\mathbf{c})} \notin \Delta_n$ such that
        1. $\Xi_n \vdash \alpha(\mathbf{c})$ (it is triggered) and
        2. Ξ_n is consistent with $\beta_1(\mathbf{c}), \ldots, \beta_m(\mathbf{c})$

      then take the next such one in the list, d, and
        - if $\Xi_n \cup \{\gamma(\mathbf{c})\}$ is consistent with each justification in $\Delta_n \cup \{d\}$ then let $\Xi_{n+1} = \Xi_n \cup \{\gamma(\mathbf{c})\}$ and $\Delta_{n+1} = \Delta_n \cup \{d\}$
        - else, abort: no extension with this ordering of defaults
    - else let $\Xi_{n+1} = \Xi_n$ and $\Delta_{n+1} = \Delta_n$
- the extension is: $\Xi = \bigcup_{i \geq 0} \Xi_i$

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instead of guessing

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to check consistency

we keep track of used defaults

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- the extension is: $\Xi = \bigcup_{i \geq 0} \Xi_i$

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the order is used

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runs may be unsuccessful

- there may be several extensions given different orderings of the ground instances in Δ
- we get the same extensions as in Reiter's approach

Alternative Approaches to Reiter's

Lukaszewicz's account

Another account Łukaszewicz (1988)

Let $\langle \Delta, \Phi \rangle$ be a default theory.

- init: $\Phi^{\star} = \Phi$

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- init: $\Phi^\star = \Phi$
- (†) take a **c**-instance of an arbitrary (unused) default $\frac{\alpha(\mathsf{x}) \ : \ \beta_1(\mathsf{x}),\ldots,\beta_m(\mathsf{x})}{\gamma(\mathsf{x})} \in \Delta$ and check:

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    1. $\Phi^\star \vdash \alpha(\mathsf{c})$ (trigger)

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  2. each $\beta_i(\mathsf{c})$ is consistent with $\Phi^\star$ (justification 1)

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  3. each justification of previously applied defaults and each $\beta_1(\mathbf{c}), \ldots, \beta_m(\mathbf{c})$ is consistent with $\Phi^\star \cup \{\gamma(\mathbf{c})\}$ (justification 2)

no reference to a guess

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  3. each justification of previously applied defaults and each $\beta_1(\mathbf{c}), \ldots, \beta_m(\mathbf{c})$ is consistent with $\Phi^\star \cup \{\gamma(\mathbf{c})\}$ (justification 2)
- if yes: $\Phi^\star = \Phi^\star \cup \{\gamma(\mathbf{c})\}$ and goto (†)

no reference
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  3. each justification of previously applied defaults and each $\beta_1(\mathbf{c}), \ldots, \beta_m(\mathbf{c})$ is consistent with $\Phi^\star \cup \{\gamma(\mathbf{c})\}$ (justification 2)
- if yes: $\Phi^\star = \Phi^\star \cup \{\gamma(\mathbf{c})\}$ and goto (†)
- if no:
  - if there is another instance of an (unused) default in Δ that wasn't tested, goto (†) and test it

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- if yes: $\Phi^\star = \Phi^\star \cup \{\gamma(\mathbf{c})\}$ and goto (†)
- if no:
  - if there is another instance of an (unused) default in Δ that wasn't tested, goto (†) and test it
  - otherwise: let $\Xi = \mathrm{Cn}(\Phi^\star)$, we found an extension.

no reference to a guess

success warranted

Some properties of the new procedure

- No guess needed.
- real procedural character
- guarantees existence of an extension
- hence: yields sometimes different results from Reiter's account

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Question

What happens in the new approach when plugging in the default theory $\langle \Delta, \Phi \rangle$ where

- $\Delta = \left\{ \frac{\alpha(x) \ : \ \beta(x) \wedge \gamma(x)}{\gamma(x)}, \frac{\gamma(x) \ : \ \neg\beta(x)}{\neg\beta(x)} \right\}$
- $\Phi = \{\alpha(c)\}$

Recall the three conditions

1. $\Phi^\star \vdash \alpha(c)$ (trigger)
2. each $\beta_i(c)$ is consistent with $\Phi^\star$ (justification 1)
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- $\Delta = \left\{ \frac{\alpha(x) \ : \ \beta(x) \land \gamma(x)}{\gamma(x)}, \frac{\gamma(x) \ : \ \neg\beta(x)}{\neg\beta(x)} \right\}$
- $\Phi = \{\alpha(\mathbf{c})\}$

- we start with $\Phi^\star = \Phi$
- test the $\mathbf{c}$-instance of the default $\frac{\alpha(x) \ : \ \beta(x) \land \gamma(x)}{\gamma(x)}$:

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- we start with $\Phi^\star = \Phi$
- test the $\mathbf{c}$-instance of the default $\frac{\alpha(x) \ : \ \beta(x) \wedge \gamma(x)}{\gamma(x)}$:
    1. $\Phi^\star \vdash \alpha(\mathbf{c})$, OK

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    1. $\Phi^\star \vdash \alpha(\mathbf{c})$, OK
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  1. $\Phi^\star \vdash \alpha(\mathbf{c})$, OK
  2. $\beta(\mathbf{c}) \wedge \gamma(\mathbf{c})$ is consistent with $\Phi^\star$, OK
  3. there are no previously used defaults, so 3 is OK
- hence, $\Phi^\star = \{\alpha(\mathbf{c}), \gamma(\mathbf{c})\}$.

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  2. $\beta(\mathbf{c}) \wedge \gamma(\mathbf{c})$ is consistent with $\Phi^\star$, OK
  3. there are no previously used defaults, so 3 is OK
- hence, $\Phi^\star = \{\alpha(\mathbf{c}), \gamma(\mathbf{c})\}$.
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  3. there are no previously used defaults, so 3 is OK
- hence, $\Phi^\star = \{\alpha(\mathbf{c}), \gamma(\mathbf{c})\}$.
- test the $\mathbf{c}$-instance of the default $\frac{\gamma(x) \;:\; \neg\beta(x)}{\neg\beta(x)}$:
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  3. there are no previously used defaults, so 3 is OK
- hence, $\Phi^\star = \{\alpha(\mathbf{c}), \gamma(\mathbf{c})\}$.
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- test the $\mathbf{c}$-instance of the default $\frac{\alpha(x)\ :\ \beta(x)\wedge\gamma(x)}{\gamma(x)}$:
  1. $\Phi^\star \vdash \alpha(\mathbf{c})$, OK
  2. $\beta(\mathbf{c}) \wedge \gamma(\mathbf{c})$ is consistent with $\Phi^\star$, OK
  3. there are no previously used defaults, so 3 is OK
- hence, $\Phi^\star = \{\alpha(\mathbf{c}), \gamma(\mathbf{c})\}$.
- test the $\mathbf{c}$-instance of the default $\frac{\gamma(x)\ :\ \neg\beta(x)}{\neg\beta(x)}$:
  1. $\Phi^\star \vdash \gamma(\mathbf{c})$, OK
  2. $\neg\beta(\mathbf{c})$ is consistent with $\Phi^\star$, OK
  3. however $\beta(\mathbf{c}) \wedge \gamma(\mathbf{c})$ is not consistent with $\Phi^\star \cup \{\neg\beta(\mathbf{c})\}$.

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  3. there are no previously used defaults, so 3 is OK
- hence, $\Phi^\star = \{\alpha(\mathbf{c}), \gamma(\mathbf{c})\}$.
- test the $\mathbf{c}$-instance of the default $\frac{\gamma(x) \ : \ \neg\beta(x)}{\neg\beta(x)}$:
  1. $\Phi^\star \vdash \gamma(\mathbf{c})$, OK
  2. $\neg\beta(\mathbf{c})$ is consistent with $\Phi^\star$, OK
  3. however $\beta(\mathbf{c}) \wedge \gamma(\mathbf{c})$ is not consistent with $\Phi^\star \cup \{\neg\beta(\mathbf{c})\}$.
- $\Xi = \mathrm{Cn}\{\alpha(\mathbf{c}), \gamma(\mathbf{c})\}$

Always having an extension is a good thing, is it?

Lukaszewicz's Fishing example

Let $T = \langle \Delta, \Phi \rangle$ where

- $\Delta = \left\{ \frac{\text{Sunday} \ : \ \text{I}-\text{go}-\text{fishing} \wedge \neg \text{I}-\text{wake}-\text{up}-\text{late}}{\text{I}-\text{go}-\text{fishing}}, \frac{\text{Holidays} \ : \ \text{I}-\text{wake}-\text{up}-\text{late}}{\text{I}-\text{wake}-\text{up}-\text{late}} \right\}$
- $\Phi = \{\text{Sunday}, \text{Holidays}\}$.

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What do you make of it?

Fixed Points and a bit of Meta-Theory

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A Fixed-Point Characterization

Non-Procedural Fixed-Point Characterizations

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Definition: Extension'

Ξ is an extension' of a default theory $\langle \Delta, \Phi \rangle$ iff it is a minimal set that satisfies the following conditions:

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Question

Is this equivalent to the procedural approach?

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Take $\langle \left\{ \frac{\top : p}{p} \right\}, \emptyset \} \rangle$.

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However, the only extension is $\mathrm{Cn}(\{p\})$. We face the

Problem of grounding

We expect that all members of the extension can be generated iteratively by chaining and detaching defaults.

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Let $\langle \Delta, \Phi \rangle$ be a default theory. Define the operator π_Φ such that for any set of formulas Γ, $\pi_\Phi(\Gamma)$ the smallest set satisfying:

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this is where Γ matters

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Definition: Extension

A set of formulas Γ is an extension of $\langle \Delta, \Phi \rangle$ iff $\pi_\Phi(\Gamma) = \Gamma$.

extensions are fixedpoints of π_Φ

OK, that's awfully complicated. Does this smallest set $\pi_\Phi(\Gamma)$ even exist for any Γ?

Proof of the existence of $\pi(\Gamma)$.

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Recall: Let $\langle \Delta, \Phi \rangle$ be a default theory. Define the operator π_Φ such that for any set of formulas Γ, $\pi_\Phi(\Gamma)$ the smallest set satisfying:

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2. Suppose $A \in \mathrm{Cn}(\Gamma')$. Hence (by monotonicity), $\Gamma'' \vdash A$ for all $\Gamma'' \in S$. Since $\mathrm{Cn}(\Gamma'') = \Gamma''$, $A \in \Gamma''$. Thus, $A \in \bigcap S$.

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3. Suppose $\alpha(\mathbf{c}) \in \Gamma'$ and $\neg\beta_i(\mathbf{c}) \notin \Gamma$ for all $i \leq n$. Hence, $\alpha(\mathbf{c}) \in \Gamma''$ for all $\Gamma'' \in S$ and thus $\gamma(\mathbf{c}) \in \Gamma''$. Thus, $\gamma(\mathbf{c}) \in \Gamma'$.

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Equivalence

Recall: Definition of Extension of $\langle \Delta, \Phi \rangle$

Ξ is an extension iff $\Xi = \mathrm{Cn}(\bigcup_{i=1}^{\infty} \Xi_i)$ where

1. $\Xi_0 = \Phi$
2. $\Xi_{i+1} = \Xi_i \cup \left\{ \gamma(\mathbf{c}) \mid \frac{\alpha(\mathbf{x}) \;:\; \beta_1(\mathbf{x}),\ldots\beta_n(\mathbf{x})}{\gamma(\mathbf{x})} \in \Delta, \Xi_i \vdash \alpha(\mathbf{c}), \neg\beta_1(\mathbf{c}), \ldots, \neg\beta_n(\mathbf{c}) \notin \Xi \right\}$

We show that Ξ is an extension of $\langle \Delta, \Phi \rangle$ iff $\pi_{\Phi}(\Xi) = \Xi$. We first observe that $\mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i)$ satisfies:

1. $\Phi \subseteq \mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i)$
2. $\mathrm{Cn}(\mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i)) = \mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i)$.
3. if $\frac{\alpha(\mathbf{c}) \;:\; \beta_1(\mathbf{c}),\ldots,\beta_n(\mathbf{c})}{\gamma(\mathbf{c})}$ is a $\mathbf{c}$-instance of some default in Δ and
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   3.2 $\neg\beta_i(\mathbf{c}) \notin \Xi$ for all $1 \leq i \leq n$ then $\gamma(\mathbf{c}) \in \mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i)$

Equivalence (2)

We show that if $\pi_{\Phi}(\Xi) = \Xi$ then Ξ is an extension (and hence $\Xi = \mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i)$).

- Since (by $(\star)$) $\pi_{\Phi}(\Xi) \subseteq \mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i)$, $\Xi \subseteq \mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i)$.

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- Let $\gamma(\mathsf{c}) \in \Xi_{i+1} \setminus \Xi_i$.
- Thus, there is a ground instance $\frac{\alpha(\mathsf{c}) \;:\; \beta_1(\mathsf{c}),\ldots,\beta_n(\mathsf{c})}{\gamma(\mathsf{c})}$ of a default in Δ such that $\Xi_i \vdash \alpha(\mathsf{c})$ and $\neg\beta_1(\mathsf{c}), \ldots \neg\beta_n(\mathsf{c}) \notin \Xi$.

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- Thus also $\alpha(\mathbf{c}) \in \pi_{\Phi}(\Xi)$ (by the inductive hypothesis).

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- By 3, $\gamma(\mathsf{c}) \in \pi_{\Phi}(\Xi) = \Xi$.

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- Altogether: $\bigcup_{i=0}^{\infty} \Xi_i \subseteq \Xi$.
- Thus, $\mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i) \subseteq \Xi$.

Equivalence (3)

We show that if Ξ is an extension (thus $\Xi = \mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i)$) then $\Xi = \pi_\Phi(\Xi)$.

- Since by $(\star)$, $\pi_\Phi(\Xi) \subseteq \mathrm{Cn}(\bigcup_{i=0}^{\infty} \Xi_i)$, also $\pi_\Phi(\Xi) \subseteq \Xi$.

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- Suppose $\Xi_i \subseteq \pi_\Phi(\Xi)$. To show $\Xi_{i+1} \subseteq \pi_\Phi(\Xi)$.

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Fixed Points and a bit of Meta-Theory

The “Cautious” Properties

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If $\langle \Delta, \Phi \rangle \vdash A$ and $\langle \Delta, \Phi \cup \{A\} \rangle \vdash B$ then $\langle \Delta, \Phi \rangle \vdash B$.

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Recall: Lemma

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- $\langle \Delta, \Phi \rangle \vdash_{\mathrm{cred}} p \vee q$.
- $\langle \Delta, \Phi \cup \{p \vee q\} \rangle \vdash_{\mathrm{cred}} \neg p$.
- But, $\langle \Delta, \Phi \rangle \nvdash_{\mathrm{cred}} \neg p$.

Default logic and monotonicity

Nonmonotonicity, both

- in the set of defaults Δ
- in the set of facts Φ

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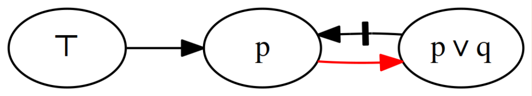
Not even cautious monotonic

Here's an example that goes back to Makinson:



- $\Delta = \left\{ \frac{\top \; : \; p}{p}, \frac{p \vee q \; : \; \neg p}{\neg p} \right\}$
- $\Phi_1 = \emptyset$
- $\Phi_2 = \{p \vee q\}$

Fixed Points and a bit of Meta-Theory

Normal Theories are quite special

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- A normal default theory always has an extension both in Reiter's and in Lukaszewicz's approach.
- For normal theories the set of Reiter extensions and the set of Lukaszewicz extensions coincides.

But, are normal defaults all we need?

Semi-normal vs. normal defaults

Compare

$$\frac{\text{has}-\text{motive}(x) \quad : \quad \text{guilty}(x) \wedge \text{suspect}(x)}{\text{suspect}(x)}$$

with

$$\frac{\text{has}-\text{motive}(x) \quad : \quad \text{guilty}(x) \wedge \text{suspect}(x)}{\text{guilty}(x) \wedge \text{suspect}(x)}$$

The expressive power of semi-normal defaults

Lukasziewicz writes: Assume, for instance, that on Sundays I usually go fishing, and suppose that you should remain agnostic about my fishing in rainy Sundays. It seems that the only appropriate representation of this situation is to use the following non-normal default:

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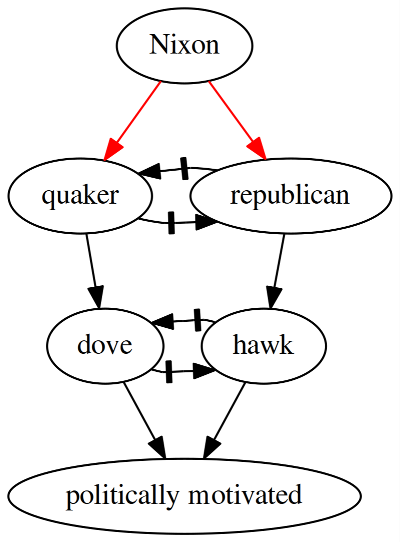
Critically evaluated this claim.

1. Why is a normal representation of this default suboptimal?
2. Do you agree with L.'s assessment that the proposed non-normal representation is adequate?

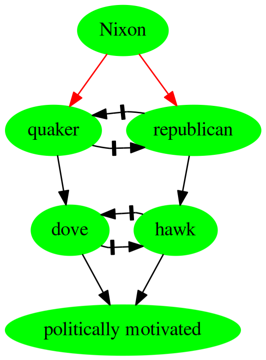
A look at various interesting examples

Floating conclusions

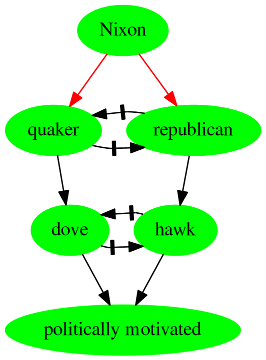


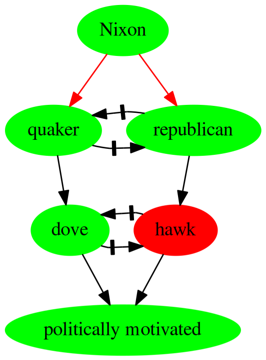
Task

1. What are the extensions of this default theory?
2. Is $\mathrm{politically{-}motivated}(\mathsf{Nixon})$ derivable?



{ Nixon, quaker, republican, dove, hawk, politically motivated }





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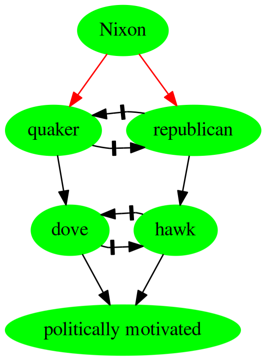

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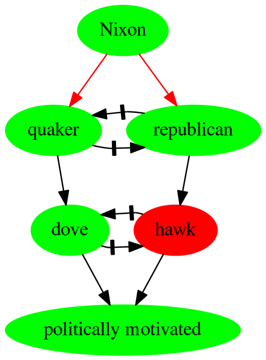

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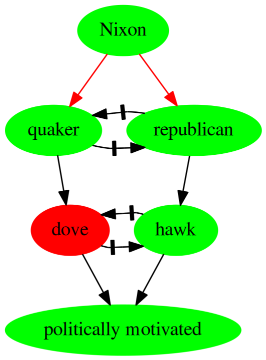

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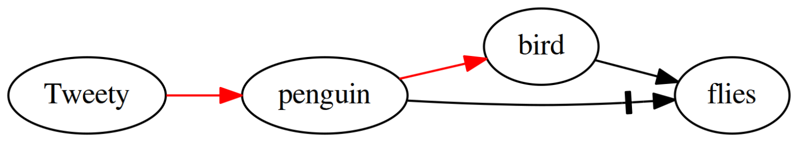

Question

Is flies(Tweety) derivable?

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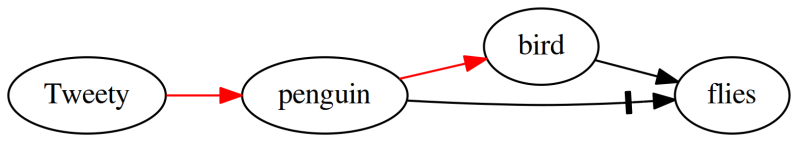

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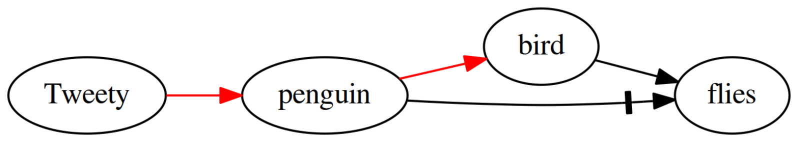

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1. one with flies(Tweety)
2. one with $\neg\mathrm{flies}$(Tweety)

Poole's Lottery Paradox

Let $T = \langle \Delta, \Phi \rangle$ where

- $\Delta = \left\{ \dfrac{\text{bird}(x) \quad : \quad \text{flies}(x) \wedge \neg\text{penguin}(x)}{\text{flies}(x) \wedge \neg\text{penguin}(x)}, \dfrac{\text{bird}(x) \quad : \quad \text{treenest}(x) \wedge \neg\text{sandpiper}(x)}{\text{treenest}(x) \wedge \neg\text{sandpiper}(x)}, \ldots \right\}$
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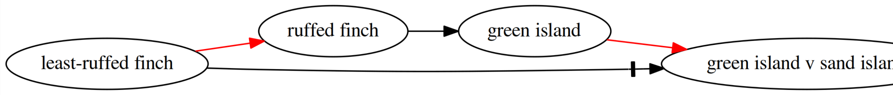
Problem

However, then we conclude $\neg\mathrm{penguin}(x) \wedge \neg\mathrm{sandpiper}(x) \wedge \neg \ldots$ for all bird-species. But then Tweety does not belong to any species of birds.

Typical birds (in an ideal sense) do not exist.

The Finch-Example



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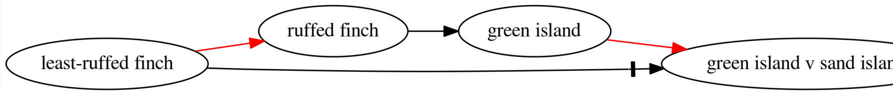

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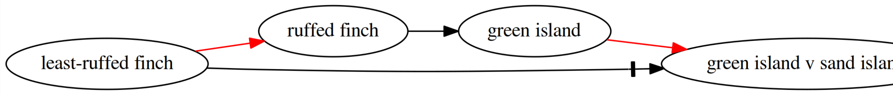

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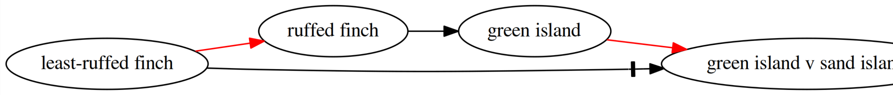

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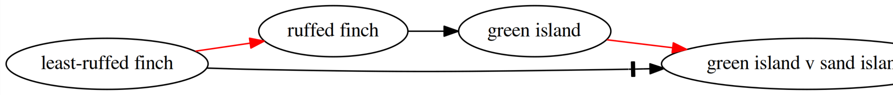

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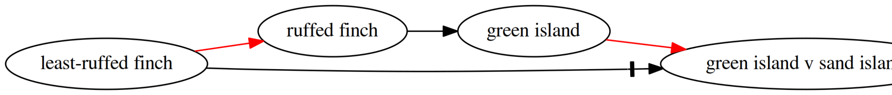

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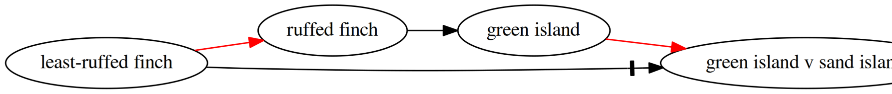
Problem

The Finch-Example



Let $T = \langle \Delta, \Phi \rangle$ where

- $\Delta = \left\{ \frac{\text{ruffed}-\text{finch}(x) \ : \ \text{green}-\text{island}(x)}{\text{green}-\text{island}(x)}, \frac{\text{least}-\text{ruffed}-\text{finch}(x) \ : \ \text{green}-\text{island}(x) \vee \text{sand}-\text{island}(x)}{\text{green}-\text{island}(x) \vee \text{sand}-\text{island}(x)} \right\}$
- Φ consists of
  - $\text{least}-\text{ruffed}-\text{finch}(\text{Frank})$
  - $\forall x(\text{least}-\text{ruffed}-\text{finch}(x) \to \text{ruffed}-\text{finch}(x))\}$

Problem

the unique extension includes both $\text{green}-\text{island}(\text{Frank})$ and $\text{green}-\text{island}(\text{Frank}) \vee \text{sand}-\text{island}(\text{Frank})$ (since both defaults are triggered)

Problems with Disjunctions

Let $T = \langle \Delta, \Phi \rangle$ where

- $\Delta = \left\{ \frac{\text{Quaker}(x) \quad : \quad \text{dove}(x)}{\text{dove}(x)}, \frac{\text{republican}(x) \quad : \quad \text{hawk}(x)}{\text{hawk}(x)} \right\}$, Let $T = \langle \Delta, \Phi \rangle$ where

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- $\Phi = \{\mathrm{Quaker}(\mathsf{Peter}) \vee \mathrm{republican}(\mathsf{Peter}),$
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- $\mathrm{hawk}(\mathsf{Peter}) \vee \mathrm{dove}(\mathsf{Peter})$,
- $\mathrm{dove}(\mathsf{Anne}) \vee \mathrm{dove}(\mathsf{George})$.

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But, in some respect this proposal is too radical:

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The good:

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- $\mathrm{flies}(\mathrm{Tweety})$

The bad:

But, in some respect this proposal is too radical:

- $\neg\mathrm{bird}(\mathrm{Keith}), \neg\mathrm{bird}(\mathrm{Fred})$
- for any ground term $t \neq \mathrm{Polly}$:
  - $\mathrm{birdsfly}(t)$
  - $\mathrm{bird}(t) \rightarrow (\mathrm{flies}(t) \wedge \neg\mathrm{baby}(t))$

Semi-normal defaults and the problem of inconsistent assumptions

Let $T = \langle \Delta, \Phi \rangle$ where

- $\Delta = \left\{ \dfrac{\text{bird}(x) \quad : \quad \text{flies}(x) \wedge \neg\text{dead}(x)}{\text{flies}(x)} \right.$
  $\left. \dfrac{\text{of}-\text{ancient}-\text{species}(x) \quad : \quad \text{fossilised}(x) \wedge \text{dead}(x)}{\text{fossilised}(x)} \right\}$

Semi-normal defaults and the problem of inconsistent assumptions

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Task

Try to see what's the problem here.

Disjunctive Default Logic

Disjunctive Default Logic

Another paradigmatic example

Suppose we have:

$$\frac{\top : \text{lh}-\text{usable} \wedge \neg \text{lh}-\text{broken}}{\text{lh}-\text{usable}} \text{ and } \frac{\top : \text{rh}-\text{usable} \wedge \neg \text{rh}-\text{broken}}{\text{rh}-\text{usable}}$$

The broken arm

Suppose we have:

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- This works fine in Reiter if we have $\Phi = \{\text{lh}-\text{broken}\}$. (Exercise: check what happens!)

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- This works fine in Reiter if we have $\Phi = \{\mathrm{lh{-}broken}\}$. (Exercise: check what happens!)
- However, if we have $\Phi_\vee = \{\mathrm{rh{-}broken} \vee \mathrm{lh{-}broken}\}$, we have a problem! (Exercise: try to see why!)

A similar formulation

We have two defaults:

$$\frac{\top : \neg \mathrm{ab}_1}{\mathrm{lh{-}usable}} \text{ and } \frac{\top : \neg \mathrm{ab}_2}{\mathrm{rh{-}usable}}$$

A similar formulation

We have two defaults:

$$\frac{\top : \neg \mathrm{ab}_1}{\mathrm{lh{-}usable}} \text{ and } \frac{\top : \neg \mathrm{ab}_2}{\mathrm{rh{-}usable}}$$

and the factual information $\Phi = \{\mathrm{lh{-}broken} \supset \mathrm{ab}_1, \mathrm{rh{-}broken} \supset \mathrm{ab}_2\} \cup \{\mathrm{lh{-}broken} \vee \mathrm{rh{-}broken}\}$.

Disjunctive Default Logic

A new disjunction to the rescue!

A new disjunction | in Gelfond et al. (1991)

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- $\Phi = \{\mathrm{lh{-}broken} \mid \mathrm{rh{-}broken}\}$.
- The new disjunction is used in such a way that in every extension it is enforced that one disjunct is true.
- Exercise: determine the extensions!

Disjunctive Defaults

The new disjunction can also appear in defaults:

$$\frac{\alpha : \beta_1, \dots, \beta_m}{\gamma_1 \mid \dots \mid \gamma_n}$$

disjunctive conclusions

where, again,

- α is the prerequisite,
- $\beta_1, \dots, \beta_m$ are the justifications, and
- $\gamma_1, \dots, \gamma_n$ are the conclusions of the default.

Disjunctive Default Theories ...

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Disjunctive Default Theories …

… consist of

- a set of disjunctive defaults and
- a set of facts (possibly with the new disjunction as the most outward connective)

Disjunctive Default Logic

What are extensions now?

Extensions of disjunctive default theories (original)

Definition 5.1 *Let D be a disjunctive default theory, and let E be a set of sentences. E is an extension for D if it is one of the minimal deductively closed sets of sentences E' satisfying the condition: For any ground instance (9) of any default from D, if $\alpha \in E'$ and $\neg\beta_1, \ldots, \neg\beta_m \notin E$ then, for some i (1øiøn), $\gamma_i \in E'$. A theorem is a sentence that belongs to all extensions.*

where 'facts' are defaults with empty justification and empty prerequisite.

- Exercise: is this problematic?

See, our slides are useful :-)

Extensions of disjunctive default theories (alternative)

Given a disjunctive default theory $\langle \Delta, \Phi \rangle$ let $\Pi_\Phi(\Gamma)$ be the operator that returns the smallest set that satisfy the following requirements:

1. for each $\alpha_1 \mid \ldots \mid \alpha_n$ in $\Pi_\Phi(\Gamma)$ there is an $i \leq n$ such that $\alpha_i \in \Pi_\Phi(\Gamma)$
2. $\Pi_\Phi(\Gamma) = Cn(\Pi_\Phi(\Gamma))$
3. for each $\frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma_1|\ldots|\gamma_m} \in \Delta$ if
    - trigger: $\alpha \in \Pi_\Phi(\Gamma)$
    - consistency: $\neg\beta_1 \notin \Gamma$ for each $i \leq n$

    then $\gamma_j \in \Pi_\Phi(\Gamma)$ for some $j \leq m$.

Γ is an extension iff $\Gamma{=}\Pi_\Phi(\Gamma)$.

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choose

Γ is an extension iff $\Gamma = \Pi_\Phi(\Gamma)$.

choose again

we want fixed points

Extensions of disjunctive default theories (alternative)

choose

- guess the extension Ξ
- init beliefs: $\Xi^\star$ pick from each $\alpha_1 \mid \ldots \mid \alpha_n \in \Phi$ a member
- (†) take a default $\frac{\alpha \ : \ \beta_1, \ldots, \beta_m}{\gamma_1 | \ldots | \gamma_n} \in \Delta$ and check whether:
  1. trigger?: $\Xi^\star \vdash \alpha$
  2. conflicted?: each β_i $(1 \leq i \leq m)$ is consistent with Ξ (!!)
- if yes: update beliefs: $\Xi^\star := \Xi^\star \cup \{\gamma_i\}$ for some $1 \leq i \leq n$
- if no:
  - try another triggered default in Δ (goto (†))
  - if there isn't: terminate.
  - if $\Xi = \mathrm{Cn}(\Xi^\star)$: extension found.

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choose

Do the two definition characterize the same extensions?

Take $\langle\{\frac{p:q}{q}, \frac{p:r}{r}\}, \{p \mid q\}\rangle$

With the operational / semi-inductive approach:

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Take $\langle \{ \frac{p:q}{q}, \frac{p:r}{r} \}, \{ p \mid q \} \rangle$

With the operational / semi-inductive approach:

We have two extensions:

1. $Cn(\{p, q, r\})$

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With the fixed point approach:

We have one extension

- namely $Cn(\{q\})$.
- note that $Cn(\{p, q, r\})$ is not an extension

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We have two extensions:

1. $Cn(\{p, q, r\})$
2. $Cn(\{q\})$

With the fixed point approach:

We have one extension

- namely $Cn(\{q\})$.
- note that $Cn(\{p, q, r\})$ is not an extension (since $\Pi_{\{p|q\}}(Cn(\{p, q, r\})) = Cn(\{q\})$)

Some examples

Compare:

$$T_1 = \left\langle \left\{ \frac{p : q}{q}, \frac{r : s}{s} \right\}, \{p \vee r\} \right\rangle$$

Some examples

Compare:

$$T_1 = \left\langle \left\{ \frac{p : q}{q}, \frac{r : s}{s} \right\}, \{p \vee r\} \right\rangle$$

with

$$T_2 = \left\langle \left\{ \frac{p : q}{q}, \frac{r : s}{s} \right\}, \{p \mid r\} \right\rangle$$

Disjunctive Default Logic

Covers

Covers: can disjunctive default theory be simulated in standard default theory?

Let a cover of a disjunctive default theory T be a Reiter default theory in which for each $\alpha_1 \mid \ldots \mid \alpha_n$ occurring in T is replaced by some α_i where $1 \leq i \leq n$.

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Example

$$T_2 = \left\langle \left\{ \frac{p : q}{q}, \frac{r : s}{s} \right\}, \{p \mid r\} \right\rangle$$

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Example

$$T_2 = \left\langle \left\{ \frac{p : q}{q}, \frac{r : s}{s} \right\}, \{p \mid r\} \right\rangle$$

We have two covers:

- $T_2' = \left\langle \left\{ \frac{p:q}{q}, \frac{r:s}{s} \right\}, \{p\} \right\rangle$

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Example

$$T_2 = \left\langle \left\{ \frac{p : q}{q}, \frac{r : s}{s} \right\}, \{p \mid r\} \right\rangle$$

We have two covers:

- $T_2' = \left\langle \left\{ \frac{p:q}{q}, \frac{r:s}{s} \right\}, \{p\} \right\rangle$
- $T_2'' = \left\langle \left\{ \frac{p:q}{q}, \frac{r:s}{s} \right\}, \{r\} \right\rangle$

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Example

$$T_2 = \left\langle \left\{ \frac{p : q}{q}, \frac{r : s}{s} \right\}, \{p \mid r\} \right\rangle$$

We have two covers:

- $T_2' = \left\langle \left\{ \frac{p:q}{q}, \frac{r:s}{s} \right\}, \{p\} \right\rangle$
- $T_2'' = \left\langle \left\{ \frac{p:q}{q}, \frac{r:s}{s} \right\}, \{r\} \right\rangle$
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- T_2' has one extension: $Cn(\{p, q\})$.
- T_2'' has one extension: $Cn(\{r, s\})$
- These exactly coincide with the extensions of T_2.

Does the set of extensions of the covers always coincide with the set of extensions of the disjunctive default theory (according to the fixed point approach or the semi-inductive approach)?

A counter-example for the fixed point approach:

Take $T_3 = \langle \{ \frac{p:q}{q}, \frac{p:r}{r} \}, \{p \mid q\} \rangle$.

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Exercise:

- determine the covers of T_3.
- determine an extension of a cover that is not a fixed point extension of T_3.

Disjunctive Default Logic

A problematic example?

A problem? A variant of the broken arm

Take $T_4 = \langle \{ \frac{\text{writing}-\text{legibly}:\neg\text{rh}-\text{broken}}{\neg\text{rh}-\text{broken}} \}, \{\text{lh}-\text{broken} \mid \text{rh}-\text{broken}, \text{writing}-\text{legibly}\} \rangle$.

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Exercise: try to see what happens and evaluate whether you find this problematic.

Disjunctive Default Logic

Some exercises

Another exercise

Let $T_5 = \langle \{ \frac{r : p \vee q}{p | q}, \frac{s : \neg p}{\neg p} \}, \{s, r\} \rangle$.

Another exercise

Let $T_5 = \langle \{ \frac{r:p\vee q}{p|q}, \frac{s:\neg p}{\neg p} \}, \{s, r\} \rangle$.

- Determine the extensions.
- Does q follow skeptically? What do you think?

Another exercise

Let $T_6 = \langle \{ \frac{p:q\vee r}{q|r}, \frac{q:s}{s}, \frac{s:v}{v}, \frac{r:v}{v}, \frac{t:\neg s}{\neg s} \}, \{p, t\} \rangle$.

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- Determine the extensions.
- Is v a skeptical consequence?
- Is $\neg s$ a skeptical consequence? What do you think about this?

Other variants

Other variants

Constrained Default Logic: relying on a consistent set of justifications

Poole's broken arm: Constrained default logic

Let
$T = \langle \{ \frac{\top : \mathrm{usable}(a) \wedge \neg \mathrm{broken}(a)}{\mathrm{usable}(a)}, \frac{\top : \mathrm{usable}(b) \wedge \neg \mathrm{broken}(b)}{\mathrm{usable}(b)} \}, \{ \mathrm{broken}(a) \vee \mathrm{broken}(b) \} \rangle$.

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- do you see why this is counter-intuitive?

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- do you see why this is counter-intuitive?
- enters: Constrained default logic (Schaub (1992))
- idea: keep track of used justifications and check whether they are consistent with the produced belief set

Fixed point characterization

Given a default theory $\langle \Delta, \Phi \rangle$ and a set of formulas Γ, let $\Pi_\Phi(\Gamma)$ be the function that returns the pair of smallest sets of formulas (Θ, Λ) that satisfies the following properties:

1. $\Phi \subseteq \Theta \subseteq \Lambda$

Fixed point characterization

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(Θ, Λ) is a constrained extension of $\langle \Delta, \Phi \rangle$ iff $\Pi_\Phi(\Lambda) = (\Theta, \Lambda)$.

Exercise

Check what happens in this approach when applied to our previous example.

Cautious Monotonicity

Some authors define variants of default logic that validate Cautious Monotonicity also by means of a refined handling of justifications. See (Brewka (1991); Antonelli (1999)).

Other variants

Introducing Priorities

First approach: ala Brewka (1994)

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- the idea is: if we have a choice between applying two triggered defaults δ and δ', we opt for the prioritized one
- a prioritized default theory is given by $\langle \Delta, \Phi, \prec \rangle$

Building extensions

Given a prioritized default theory $\langle \Delta, \Phi, \prec \rangle$ we build its extension as follows:

- add all facts to the initial belief set: $\Xi^\star = \Phi$
- let $\Delta^\star = \Delta$
- loop:
  - check if there is a smallest $\frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in \Delta^\star$ that is
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  - if yes: let $\Xi^\star := \Xi^\star \cup \{\gamma\}$ and $\Delta^\star := \Delta^\star - \frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma}$
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- let $T = \langle\{\delta_1 = \frac{b:f}{f}, \delta_2 = \frac{p:\neg f}{\neg f}\}, \{p, p \rightarrow b\}, \{(\delta_2, \delta_1)\}\rangle$. What can you derive?

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Exercise:

let $\prec$ be a non-linear strict order on $\Delta = \{\delta_1, \delta_2, \delta_3\}$ for which $\delta_1 \prec \delta_2$ and $\delta_1 \prec \delta_3$. Find all linear completions of $\prec$.

An Alternative Recent Approach ...

... you find in (Horty (2007, 2012)).

A Semantics for Default Logic

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Basic Idea following (Lin and Shoham (1990, 1992))

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- K for knowledge (as in computer science, not as in philosophy)

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General Translation of Defaults

$$\frac{A : B_1, \ldots, B_n}{C} \rightsquigarrow \mathsf{K}A \wedge \neg\mathbf{A}\neg B_1 \wedge \ldots \wedge \neg\mathbf{A}\neg B_n \supset \mathsf{K}C$$

Shoham is a co-author, so let's see the semantic selection!

Going nonmonotonic: Selection Semantics

We define the following order on the models of our logic:

Definition 1 (Ordering)

Where for any model M, $\mathsf{K}(M) = \{B \mid M \models \mathsf{K}B\}$ and $\mathsf{A}(M) = \{B \mid M \models \mathsf{A}B\}$,
M is preferred over M', written $M \sqsubset M'$, iff

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A Semantics for Default Logic

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- Take a model with $\neg\mathsf{A}p$. But then also $\mathsf{K}p$ holds, and thus the model is not selected.

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- In view of the first disjunction, we have two types of selected models.
    1. one in which $\mathsf{K}p$ and $\neg\mathsf{K}q$. Thus, also $\mathsf{K}r$

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- We can even deal with disjunctions in our rich language.
- Take $\{\mathsf{K}p \vee \mathsf{K}q, \mathsf{K}p \wedge \neg\mathsf{A}\neg r \supset \mathsf{K}r, \mathsf{K}q \wedge \neg\mathsf{A}\neg \supset \mathsf{K}r\}$.
- In view of the first disjunction, we have two types of selected models.
    1. one in which $\mathsf{K}p$ and $\neg\mathsf{K}q$. Thus, also $\mathsf{K}r$
    2. one in which $\mathsf{K}q$ and $\neg\mathsf{K}p$. Thus, also $\mathsf{K}r$

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