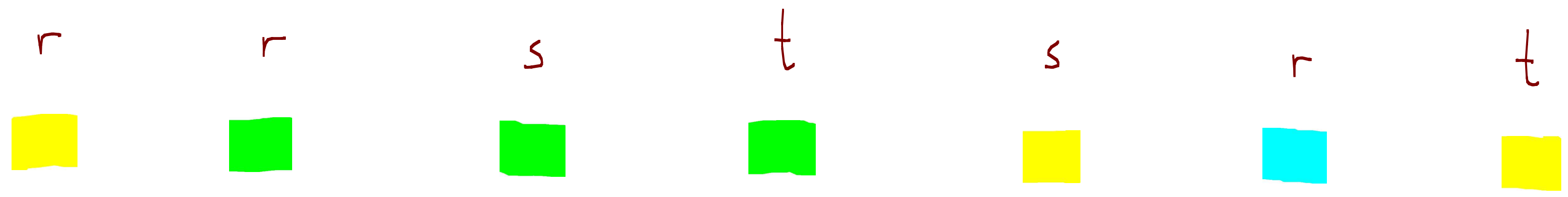
d a t a w o r d s

First-order logic

$$\forall x \; \big(r(x) \Rightarrow \exists y \; (x < y \wedge s(y) \wedge x \sim y) \big)$$

$$\forall x \; \big(\neg r(x) \Rightarrow \exists y \; (y < x \wedge y \sim x) \big)$$

$$\forall x \; \big(s(x) \Rightarrow \exists y \; (x < y \wedge t(y) \wedge x \sim y \wedge \forall z \; (x < z < y \Rightarrow r(z))) \big)$$



DECIDABILITY OF SECOND-ORDER THEORIES AND AUTOMATA ON INFINITE TREES[1]

BY

MICHAEL O. RABIN

Introduction. In this paper we solve the decision problem of a certain second-order mathematical theory and apply it to obtain a large number of decidability results. The method of solution involves the development of a theory of automata on infinite trees—a chapter in combinatorial mathematics which may be of independent interest.

Let $\Sigma=\{0, 1\}$, and denote by T the set of all words (finite sequences) on Σ. Let $r_0: T\to T$ and $r_1: T\to T$ be, respectively, the *successor functions* $r_0(x)=x0$ and $r_1(x)=x1$, $x\in T$. Our main result is that *the* (*monadic*) *second-order theory of the structure* $\langle T, r_0, r_1\rangle$ *of two successor functions is decidable*. This answers a question raised by Büchi [1].

It turns out that this result is very powerful and many difficult decidability results follow from it by simple reductions. The decision procedures obtained by this method are elementary recursive (in the sense of Kalmar). The applications include the following. (Whenever we refer, in this paper, to second-order theories, we mean monadic second-order; weak second-order means quantification restricted to finite subsets of the domain.)

The second-order theory of countable linearly ordered sets is proved decidable. As a corollary we get that the weak second-order theory of arbitrary linearly ordered sets is decidable; a result due to Läuchli [9] which improves on a result of Ehrenfeucht [5].

In [4] Ehrenfeucht announced the decidability of the first-order theory of a unary function. We prove that the second-order theory of a unary function with a countable domain is decidable. Also, the weak second-order theory of a unary function with an arbitrary domain is decidable.

There are also applications to point set topology. Let CD be Cantor's discontinuum (i.e., $\{0, 1\}^{\omega}$ with the product topology). Let F_{σ} be the lattice of all subsets of CD which are denumerable unions of closed sets, and let L_c be the sublattice of all closed subsets of CD. The first-order theory of the lattice F_{σ}, with L_c as a distinguished sublattice, is decidable. Similar results hold for the real line with the usual topology. This answers in the affirmative Grzegorczyk's question [8] whether

Presented to the Society, July 5, 1967; received by the editors March 4, 1968.

(1) The author wishes to thank S. Winograd for many extremely helpful conversations on the topics of this paper. Preparation of this paper was supported, in part, by the U.S. Office of Naval Research, Information Systems Branch, under Contract F61052 67 0055.

Satisfiability decidable ?

$FO^2(\sim, <, +1)$

$FO^3(\sim, +1)$ — no

$FO^2(\prec, <, +1)$ — no

$FO^2(\sim_1, \sim_2, <, +1)$ — no

[Bojańczyk, David, Muscholl, Schwentick, Segoufin ToCL '11]

[Björklund & Bojańczyk MFCS '07]

$FO^2(\sim, <, +1)$

$\Big\downarrow$ lin. time e.g. [Grädel & Otto TCS '99]

Scott normal form

$$\exists R_1 \cdots \exists R_n \left(\forall x \forall y\, \chi \wedge \bigwedge_i \forall x \exists y\, \chi_i \right)$$

quantifier-free

$FO^2(\sim, <, +1)$

$\downarrow$ lin. time e.g. [Grädel & Otto TCS '99]

Scott normal form

$$\exists R_1 \cdots \exists R_n \left(\underbrace{\forall x \forall y\, \chi}_{} \wedge \bigwedge_i \underbrace{\forall x \exists y\, \chi_i}_{} \right)$$

quantifier-free

... exp. time [Bojańczyk et al. ToCL '11]

$$\bigwedge_i \forall x \forall y \Big(\big(\alpha(x) \wedge \beta(y) \wedge \delta(x,y) \big) \rightarrow \gamma(x,y) \Big)$$

types

$x \sim y$

$x \not\sim y$

$<, +1$

clause

$FO^2(\sim, <, +1)$

lin. time

e.g. [Grädel & Otto TCS '99]

Scott normal form

$$\exists R_1 \cdots \exists R_n \left(\underbrace{\forall x \forall y\, \chi}_{} \wedge \bigwedge_i \underbrace{\forall x \exists y\, \chi_i}_{} \right)$$

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... exp. time

[Bojańczyk et al. ToCL '11]

2-exp. time

$$\bigwedge_i \forall x \forall y \Big(\big(\alpha(x) \wedge \beta(y) \wedge \delta(x,y) \big) \rightarrow \gamma(x,y) \Big)$$

types

$x \sim y$

$x \not\sim y$

$<, +1$

clause

data automata

$FO^2(\sim, <, +1)$

$\downarrow$ lin. time   e.g. [Grädel & Otto TCS '99]

Scott normal form

$$\exists R_1 \cdots \exists R_n \Big(\underbrace{\forall x \forall y\, \chi}_{} \wedge \bigwedge_i \underbrace{\forall x \exists y\, \chi_i}_{} \Big)$$

quantifier-free

$\searrow$ ... exp. time   [Bojańczyk et al. ToCL '11]

2-exp. time

$$\bigwedge_i \forall x \forall y \Big(\big(\alpha(x) \wedge \beta(y) \wedge \delta(x,y) \big) \to \gamma(x,y) \Big)$$

types $x \sim y$, $x \not\sim y$ $<, +1$ clause

class-memory automata

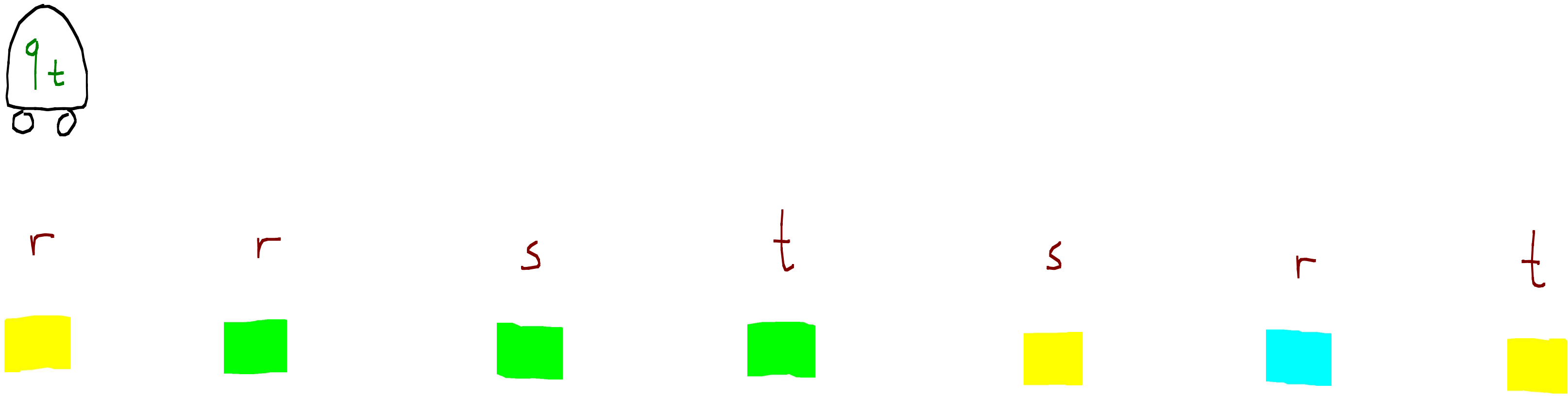
$$\left\langle Q, \Sigma, \delta, q_I, F_G, F_L \right\rangle$$

$\delta \subseteq Q \times \Sigma \times (Q \cup \{\perp\}) \times Q$

$F_G \subseteq F_L$

[Björklund & Schwentick TCS '10]

'For each datum, its subword is in $(r\ s\ t)^*$.'



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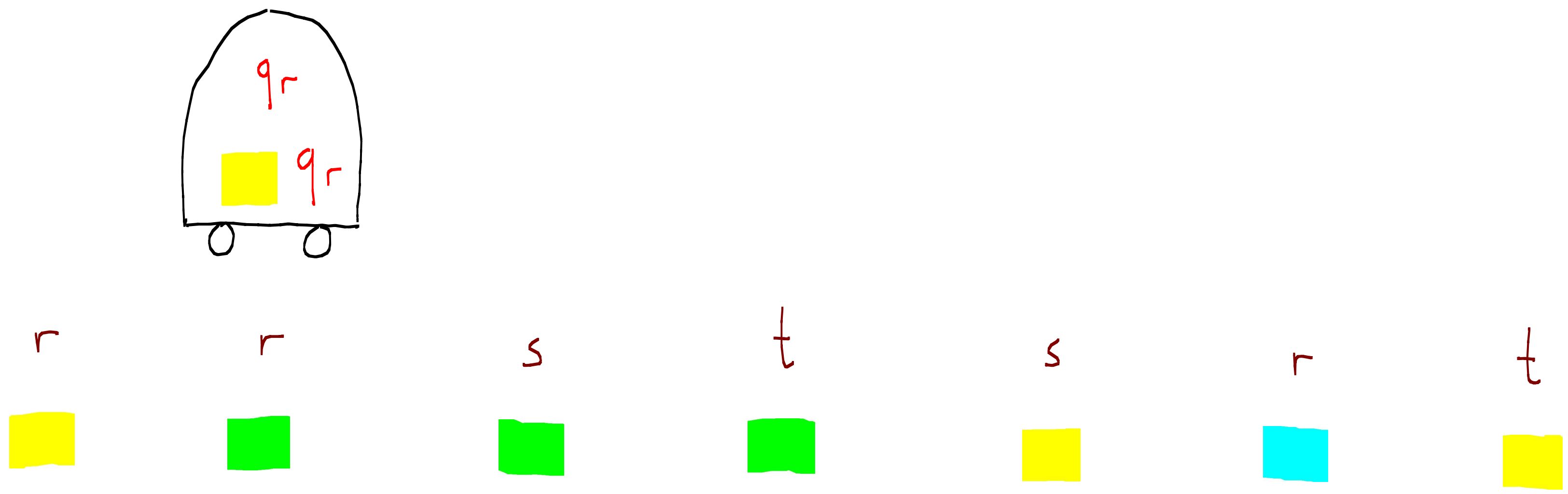

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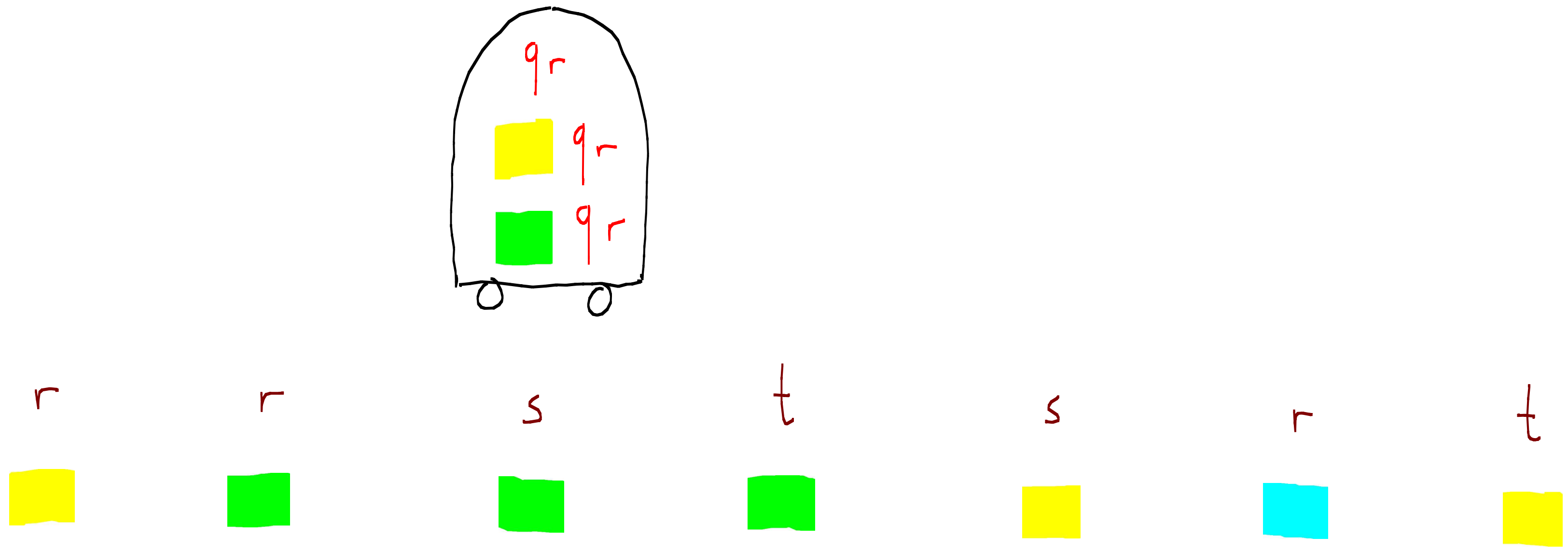

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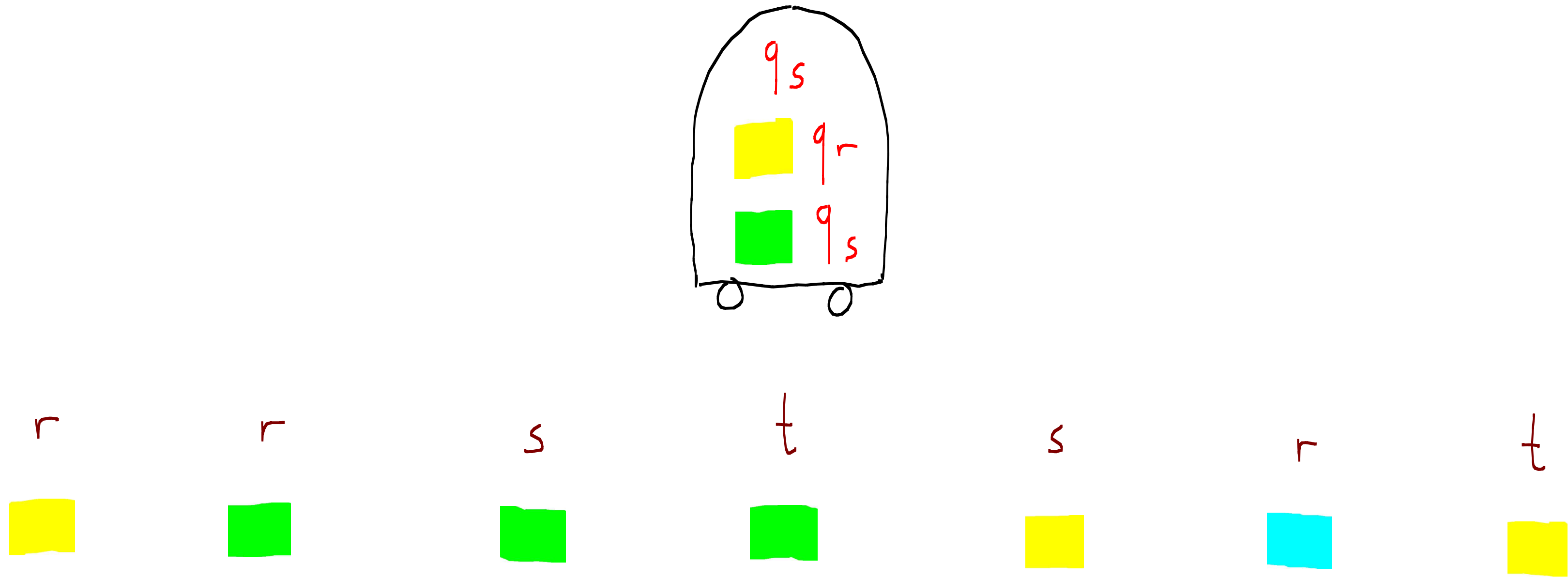

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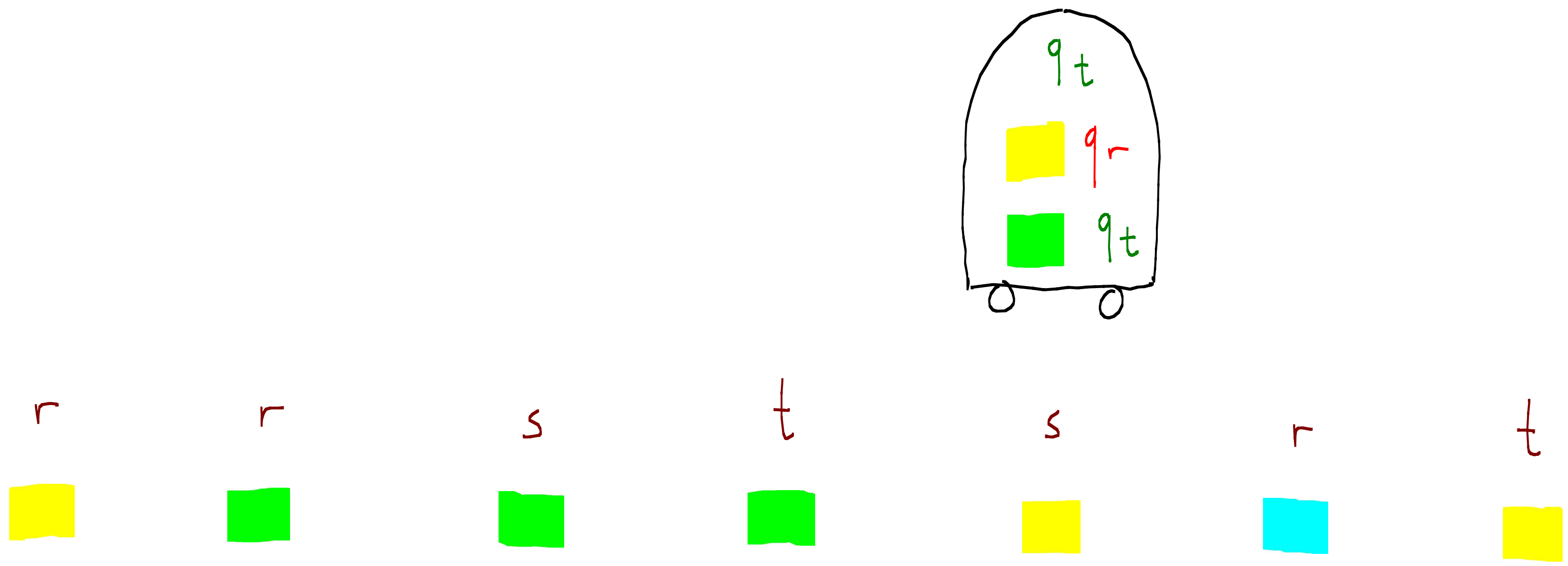

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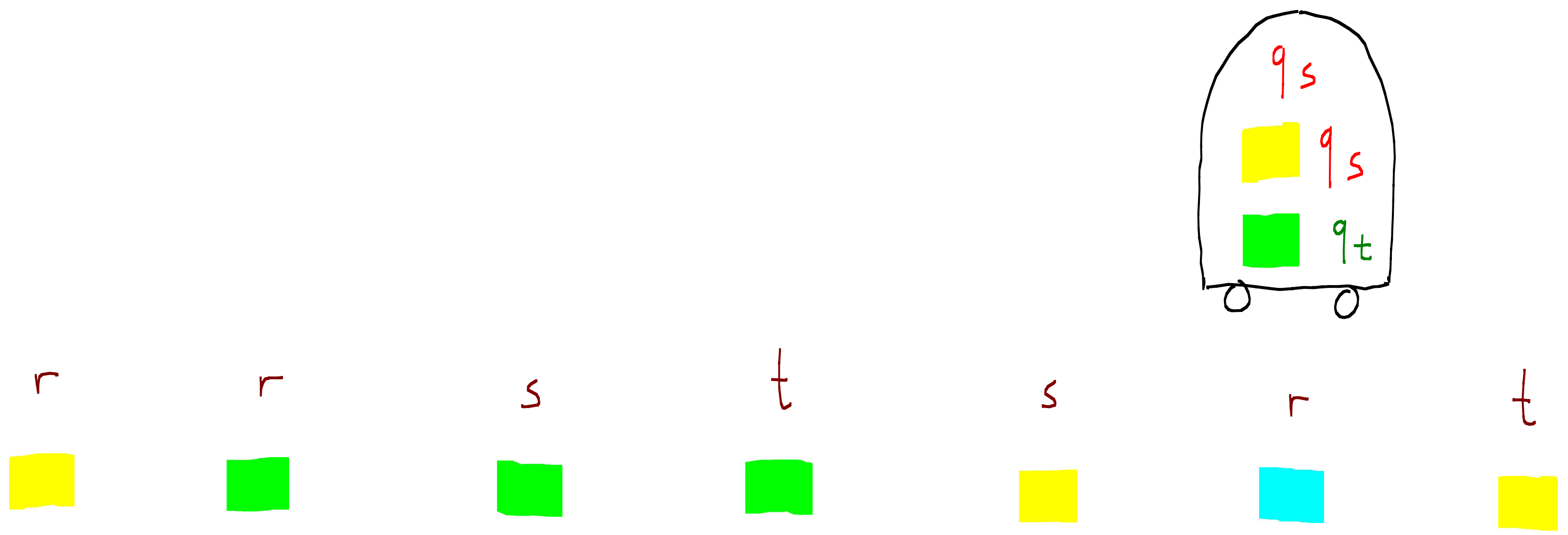

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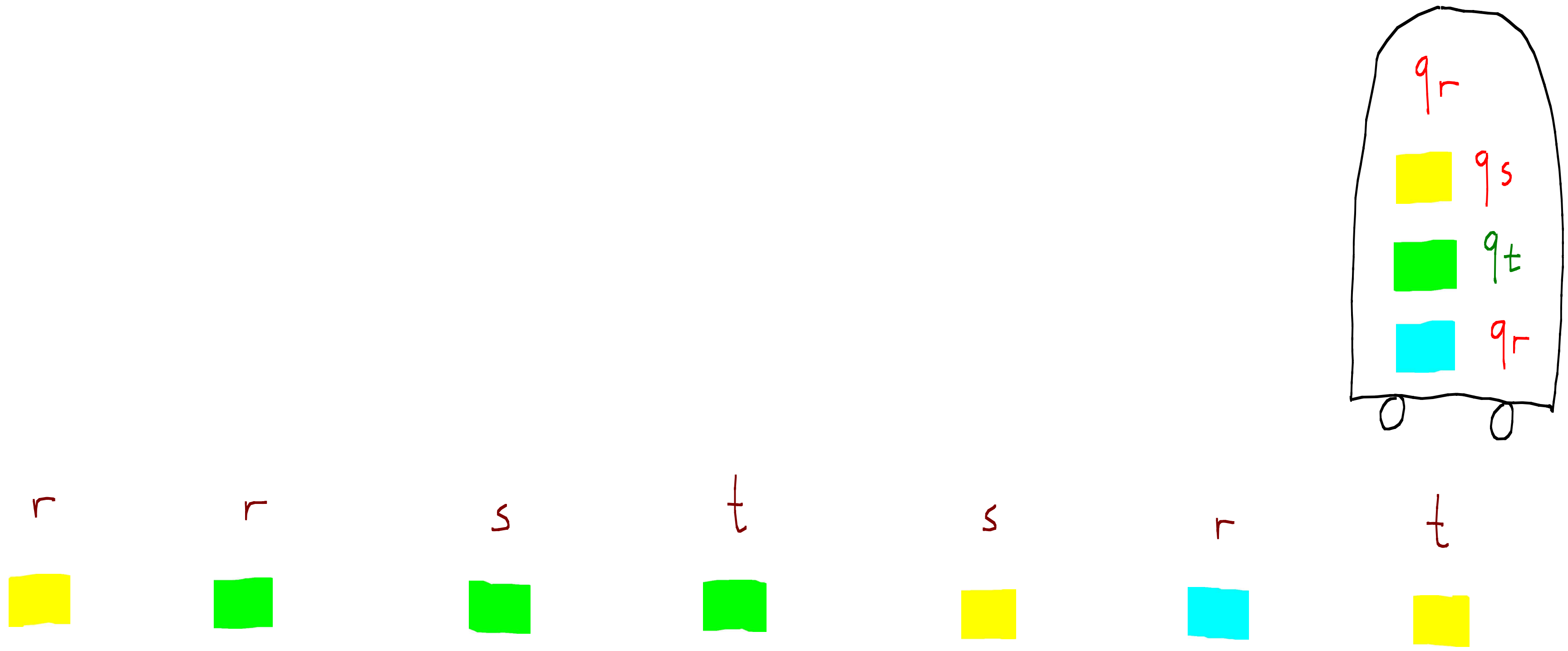

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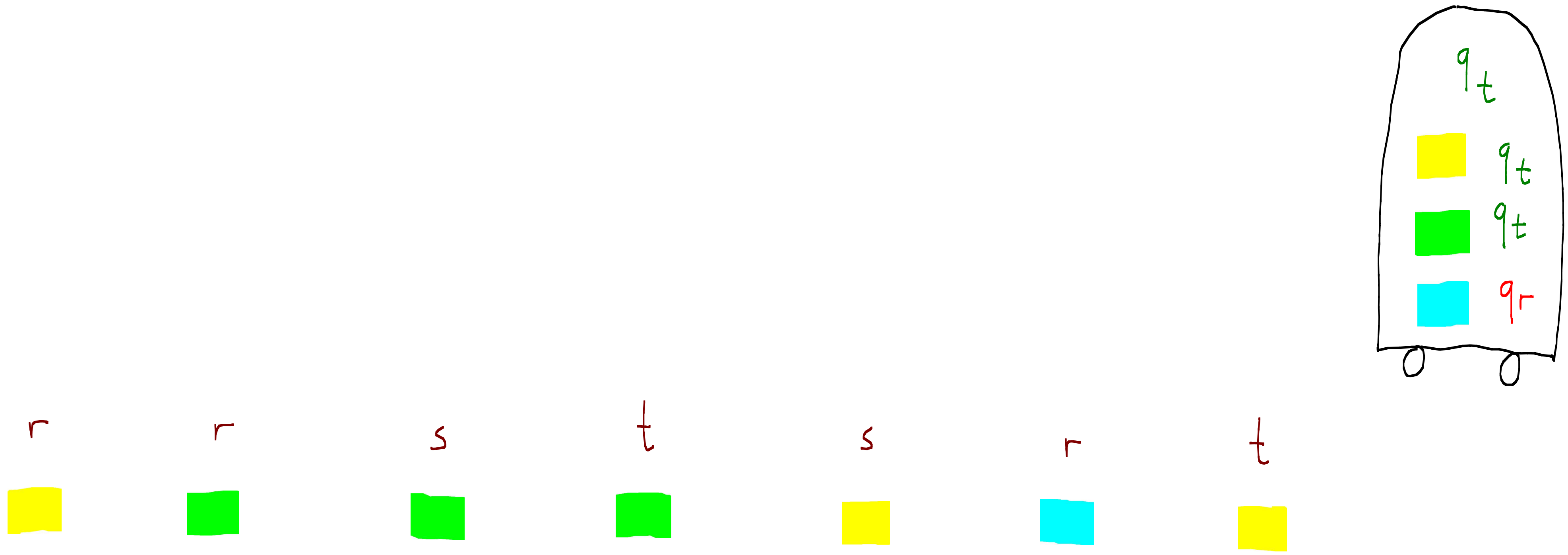


Logical characterisation of CMA

As expressive as $\mathrm{EMSO}^2(\sim, \not+1, <, +1)$.

Closure properties of CMA

intersection ✓

union ✓

complement ✗

renaming ✓

[Bojańczyk et al. ToCL '11]

[Björklund & Schwentick TCS '10]

class-memory automata

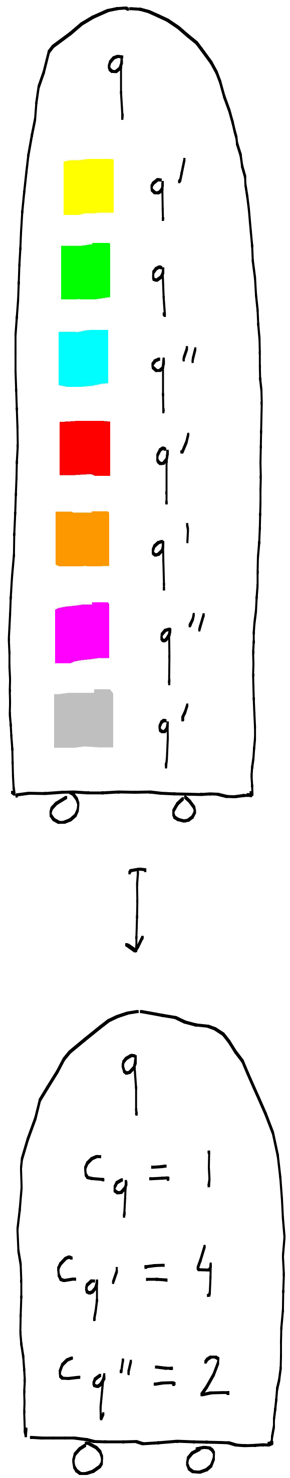
$\Big\downarrow$ log. space [Gischer C. ACM '81]

Minsky automata

class-memory automata

$\Big\downarrow$ log. space [Gischer C. ACM '81]

Minsky automata



class-memory automata

$\downarrow$ log. space [Gischer C. ACM '81]

counter automata

$$\langle Q, \Sigma, C, \delta, q_I, F_Q, F_C \rangle$$

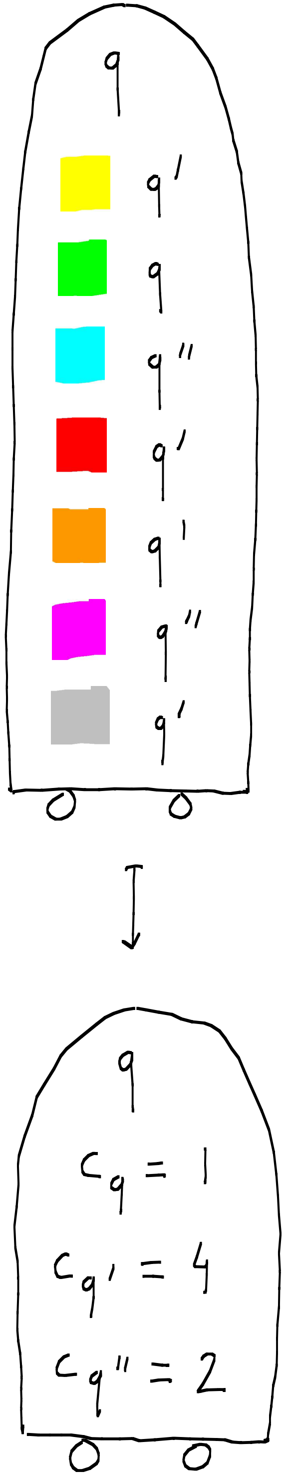
$$\delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times \{inc, dec\} \times C \times Q$$

$$F_Q \subseteq Q$$

$$F_C \subseteq C$$

Nonemptiness decidable [Mayr SIAM JoC '84, Kosaraju STOC '82]

ExpSpace-hard [Lipton '76]



The Complexity of Reachability in Vector Addition Systems

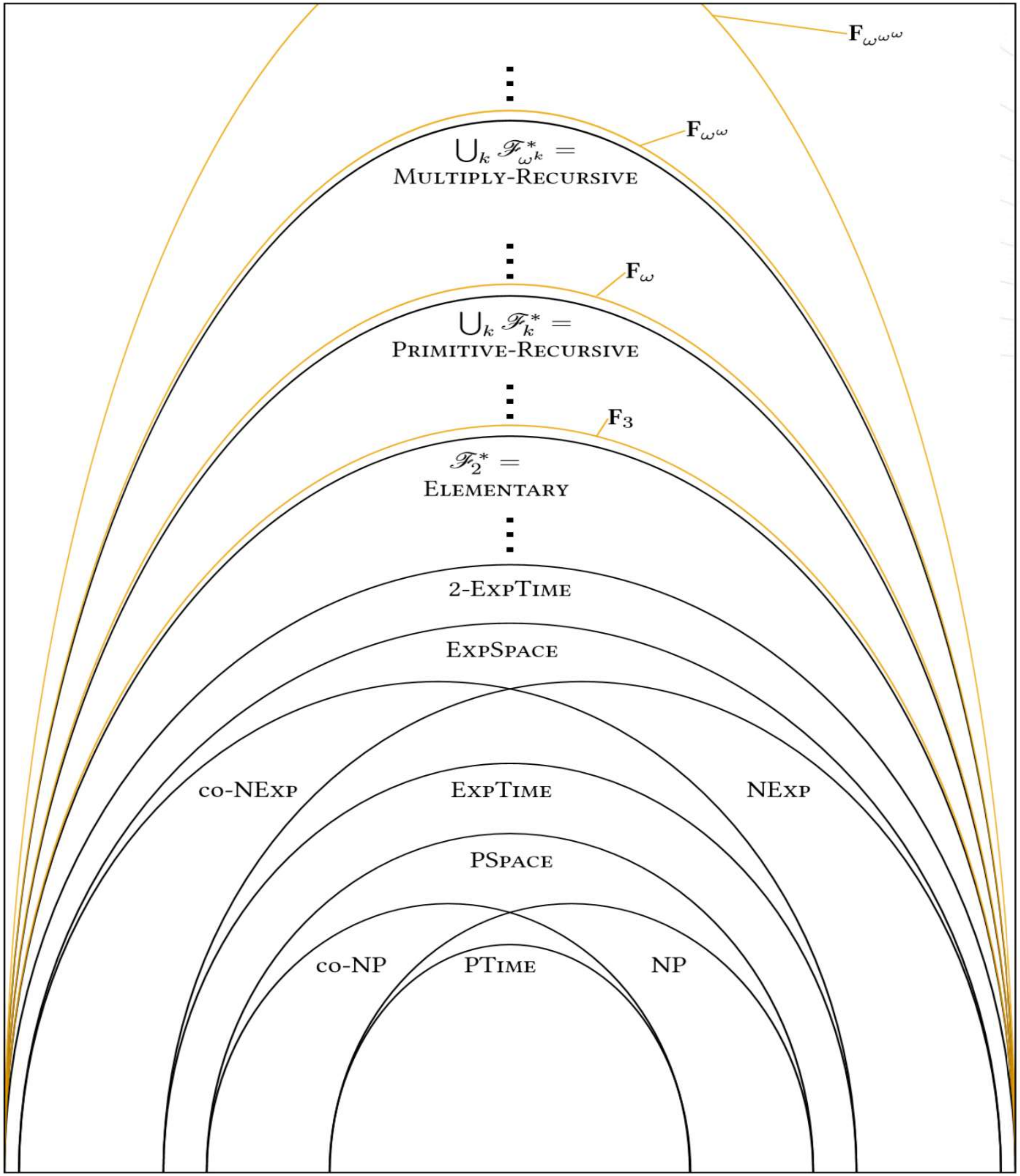
SYLVAIN SCHMITZ,
LSV, ENS Cachan & CNRS & INRIA, Université Paris-Saclay

The program of the 30th Symposium on Logic in Computer Science held in 2015 in Kyoto included two contributions on the computational complexity of the *reachability problem* for vector addition systems: Blondin, Finkel, Göller, Haase, and McKenzie [2015] attacked the problem by providing the first tight complexity bounds in the case of dimension 2 systems with states, while Leroux and Schmitz [2015] proved the first complexity upper bound in the general case. The purpose of this column is to present the main ideas behind these two results, and more generally survey the current state of affairs.

1. INTRODUCTION

Vector addition systems with states (VASS), or equivalently Petri nets, find a wide range of applications in the modelling of concurrent, chemical, biological, or business processes. Maybe more importantly for this column, their algorithmics, and in particular the decidability of their *reachability problem* [Mayr 1981; Kosaraju 1982; Lambert 1992; Leroux 2011], is the cornerstone of many decidability results in logic, automata, verification, etc.—see Section 5 for a few examples.

In spite of its importance, fairly little is known about the computational complexity of the reachability problem. Regarding the general case, the inclusive surveys on the complexity of decision problems on VASS by Esparza and Nielsen [1994] and Esparza [1998] could only point to the EXPSPACE lower bound of Lipton [1976] and to the fact that the running time of the known algorithms is not primitive recursive: *no* complexity upper bound was known, besides decidability first proven in 1981 by Mayr. When turning to restricted versions of the problem, the 2-dimensional case was only known to be in 2-EXP [Howell, Rosier, Huynh, and Yen 1986] and NP-hard [Rosier and Yen 1986].



$\mathbf{F}_{\omega^{\omega^\omega}}$
$\mathbf{F}_{\omega^\omega}$
$\bigcup_k \mathscr{F}^*_{\omega^k} =$
MULTIPLY-RECURSIVE
$\mathbf{F}_\omega$
$\bigcup_k \mathscr{F}^*_k =$
PRIMITIVE-RECURSIVE
$\mathbf{F}_3$
$\mathscr{F}^*_2 =$
ELEMENTARY
2-EXPTIME
EXPSPACE
co-NEXP
EXPTIME
NEXP
PSPACE
co-NP
PTIME
NP

An EXPSPACE Lower Bound

APRIL 8, 2009

by rjlipton

tags: Algorithms, decision procedure, language, Machine, Meyer, Petri Nets, Problems, Simulation, Turing, vector addition system

Getting The Result

The proof of the lower bound result for VAS reachability started with a conversation with Meyer at a STOC conference. He told me that he had just proved, with Paterson, that the n-dimensional VAS reachability problem required at least PSPACE. I had been thinking about VAS's and realized instantly that I could prove the same result with fewer dimensions. I knew a couple of tricks that Albert did not, so I played it cool and said nothing to Albert. But once back at home–Yale at the time–I worked hard until I had a proof of the $\mathsf{EXPSPACE}$ lower bound.

I then called Albert to tell him the news. It was a strange phone call. He did not believe that I could prove such a theorem. The conversation consisted of Albert listing a variety of mistakes that I must have made. I kept saying no I did not do that, or no I realize that. Finally, the call ended without me saying anything to him about the proof.

Open Problems

The main open problem is to improve the lower bound on the reachability problem for VAS's. There is currently a huge gap between the lower bound of $\mathsf{EXPSPACE}$ and the upper bound, which is not even primitive recursive. The current construction of the lower bound is tight, and I see no way to improve it. But, perhaps you will see something that we all have missed.

$$\begin{array}{l}\text{counter automata} \\ \Big\downarrow \text{ log. space} \qquad \text{[Bojańczyk et al. ToCL'11]} \\ FO^2(\sim, <, +1)\end{array}$$

$$
\begin{tikzcd}
\text{counter automata} \arrow[d, "\text{log. space}"] \\
FO^2(\sim, <, +1)
\end{tikzcd}
$$

[Bojańczyk et al. ToCL'11]

$$
q_t \xrightarrow[\text{inc}(c_r)]{r} q_r \xrightarrow[\text{inc}(c_r)]{r} q_r \xrightarrow[\text{dec}(c_r)]{s} q_s' \xrightarrow[\text{inc}(c_s)]{\varepsilon} q_s \quad \cdots
$$

counter automata

$$\Big\downarrow \text{ log. space}$$

[Bojańczyk et al. ToCL '11]

$FO^2(\sim, <, +1)$

$$q_t \xrightarrow[\mathrm{inc}(c_r)]{r} q_r \xrightarrow[\mathrm{inc}(c_r)]{r} q_r \xrightarrow[\mathrm{dec}(c_r)]{s} q_s' \xrightarrow[\mathrm{inc}(c_s)]{\varepsilon} q_s \quad \cdots$$

counter automata

$\big\downarrow$ log. space [Bojańczyk et al. ToCL'11]

$FO^2(\sim, <, +1)$

$$q_t \xrightarrow[\mathrm{inc}(c_r)]{r} q_r \xrightarrow[\mathrm{inc}(c_r)]{r} q_r \xrightarrow[\mathrm{dec}(c_r)]{s} q'_s \xrightarrow[\mathrm{inc}(c_s)]{\varepsilon} q_s \quad \cdots$$

$$\bigwedge_{\substack{q,q',q'',q''' \in Q \\ a,a' \in \Sigma \cup \{\varepsilon\} \\ c \in C}} \forall x \forall y \, \neg \big(\langle q, a, \mathrm{inc}, c, q' \rangle (x) \wedge \langle q'', a', \mathrm{inc}, c, q''' \rangle (y) \wedge x < y \wedge x \sim y \big)$$

$$\begin{array}{c} \text{counter automata} \\ \Big\downarrow \text{ log. space} \\ FO^2(\sim, <, +1) \end{array}$$

[Bojańczyk et al. ToCL'11]

$$q_t \xrightarrow[inc(c_r)]{r} q_r \xrightarrow[inc(c_r)]{r} q_r \xrightarrow[dec(c_r)]{s} q'_s \xrightarrow[inc(c_s)]{\varepsilon} q_s \quad \cdots$$

$$\bigwedge_{\substack{q,q',q'',q''' \in Q \\ a,a' \in \Sigma \cup \{\varepsilon\} \\ c \in C}} \forall x \forall y \neg \big(\langle q, a, inc, c, q' \rangle (x) \wedge \langle q'', a', inc, c, q''' \rangle (y) \wedge x < y \wedge x \sim y \big)$$

$$\bigwedge_{\substack{q,q' \in Q \\ a \in \Sigma \cup \{\varepsilon\} \\ c \in C}} \forall x \Big(\langle q, a, dec, c, q' \rangle (x) \Rightarrow \bigvee_{\substack{q'',q''' \in Q \\ a' \in \Sigma \cup \{\varepsilon\}}} \exists y \big(\langle q'', a', inc, c, q''' \rangle (y) \wedge y < x \wedge y \sim x \big) \Big)$$

Corollary

$$\left\{ f(\mathrm{str}(L(\varphi))) \;:\; \begin{array}{l} \varphi \text{ in } FO^2(\sim, <, +1) \\ f \text{ homomorphism} \end{array} \right\}$$

$$=$$

$$\left\{ f(\mathrm{str}(L(\mathcal{A}))) \;:\; \begin{array}{l} \mathcal{A} \text{ a CMA} \\ f \text{ homomorphism} \end{array} \right\}$$

$$=$$

$$\left\{ L(\mathcal{C}) \;:\; \mathcal{C} \text{ a CA} \right\}$$

ω-Satisfiability

$FO_2(\sim, <, +1)$

$\downarrow$

class-memory ω-automata

$$\left\langle \begin{array}{c} Q,\ \Sigma,\ \delta, \\ q_I,\ F_G,\ F_L \end{array} \right\rangle$$

F_G occurs inf. often and for every datum, F_L occurs inf. often

Lemma

[Dickson AJM 1913]

On any $\mathbb{N}^k$,

the product $\leq$

is a wqo.

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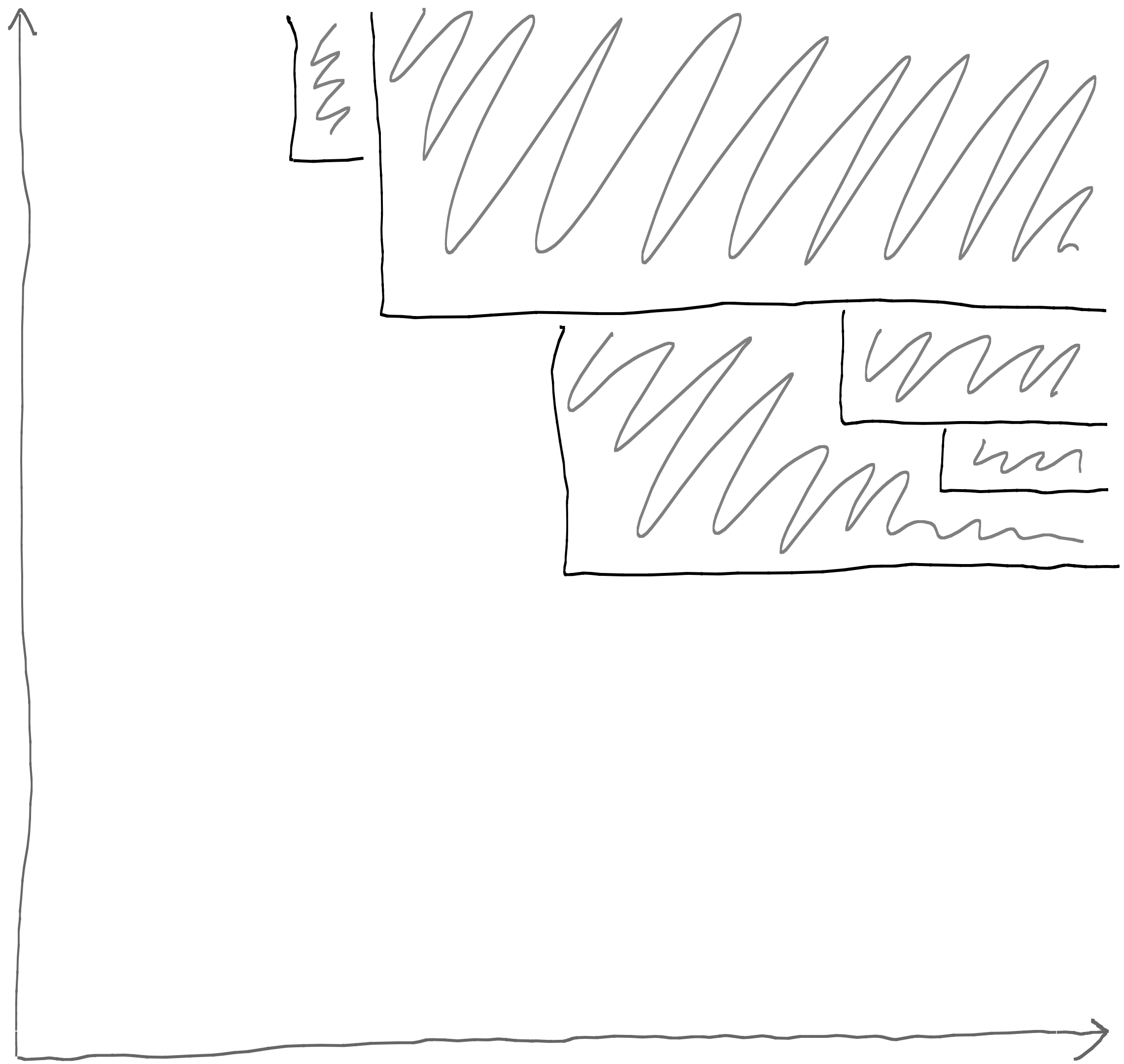

Lemma

[Dickson AJM 1913]

On any $\mathbb{N}^k$, the product $\leq$ is a wqo.

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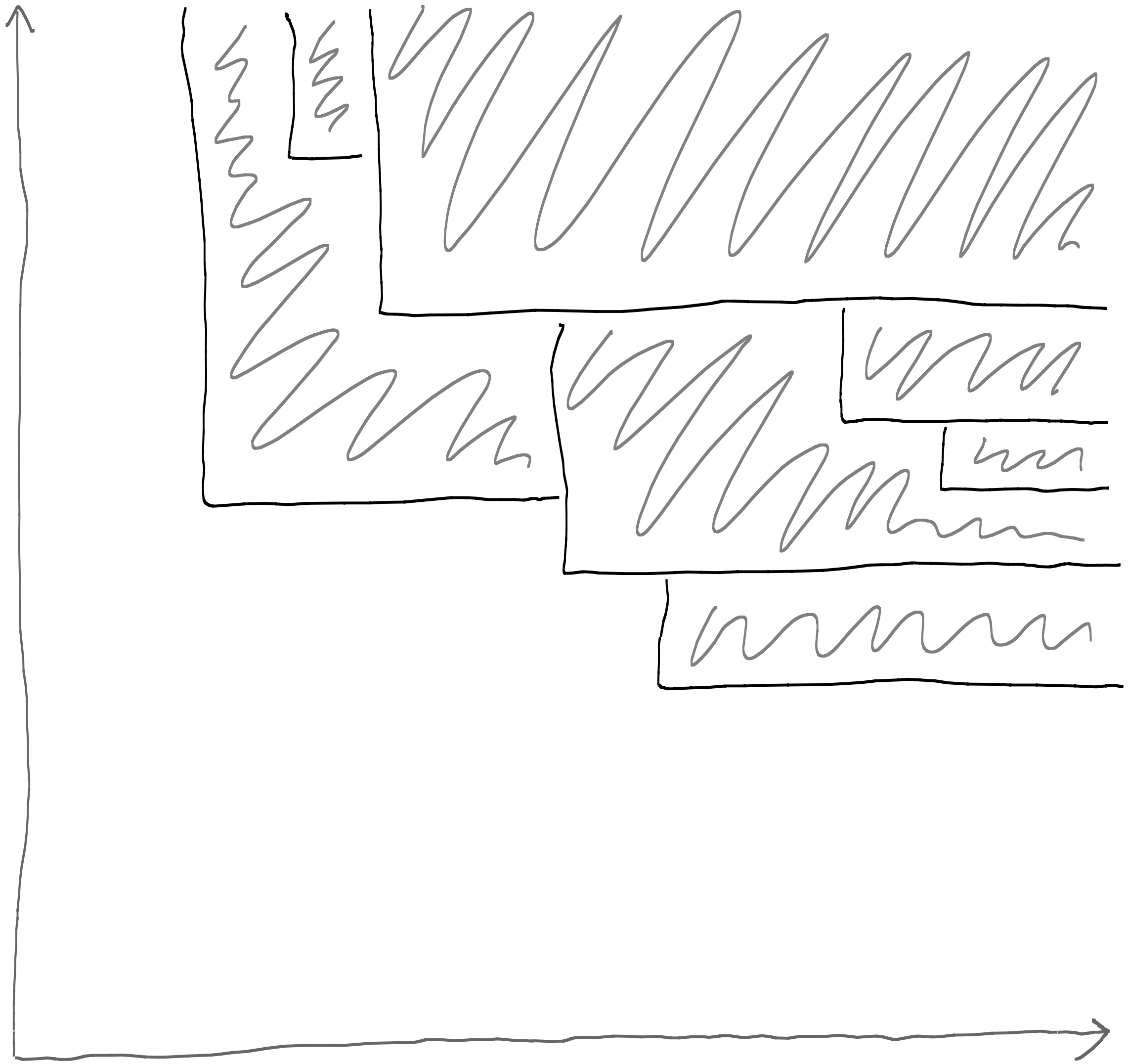

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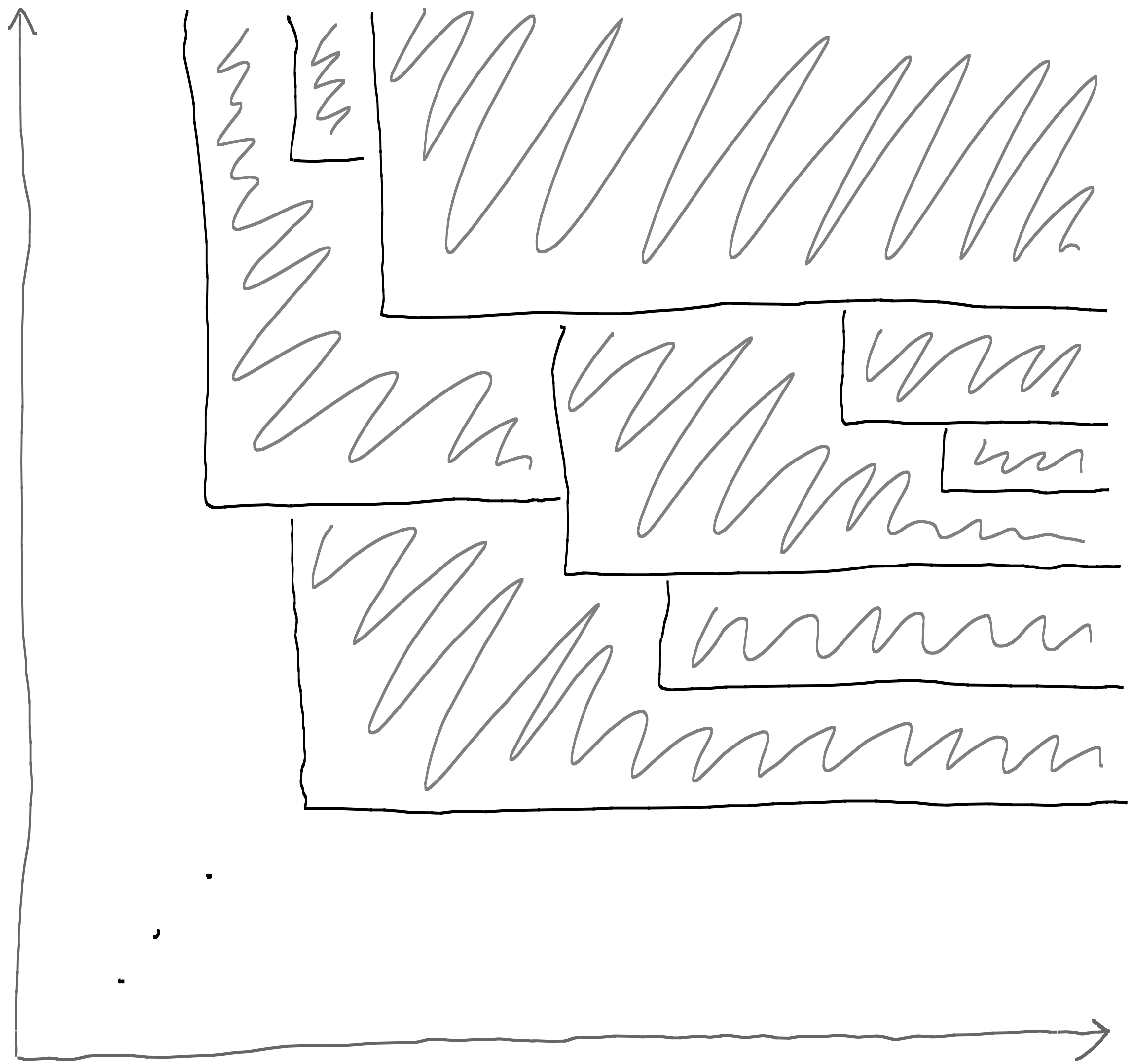

Lemma

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On any $\mathbb{N}^k$,

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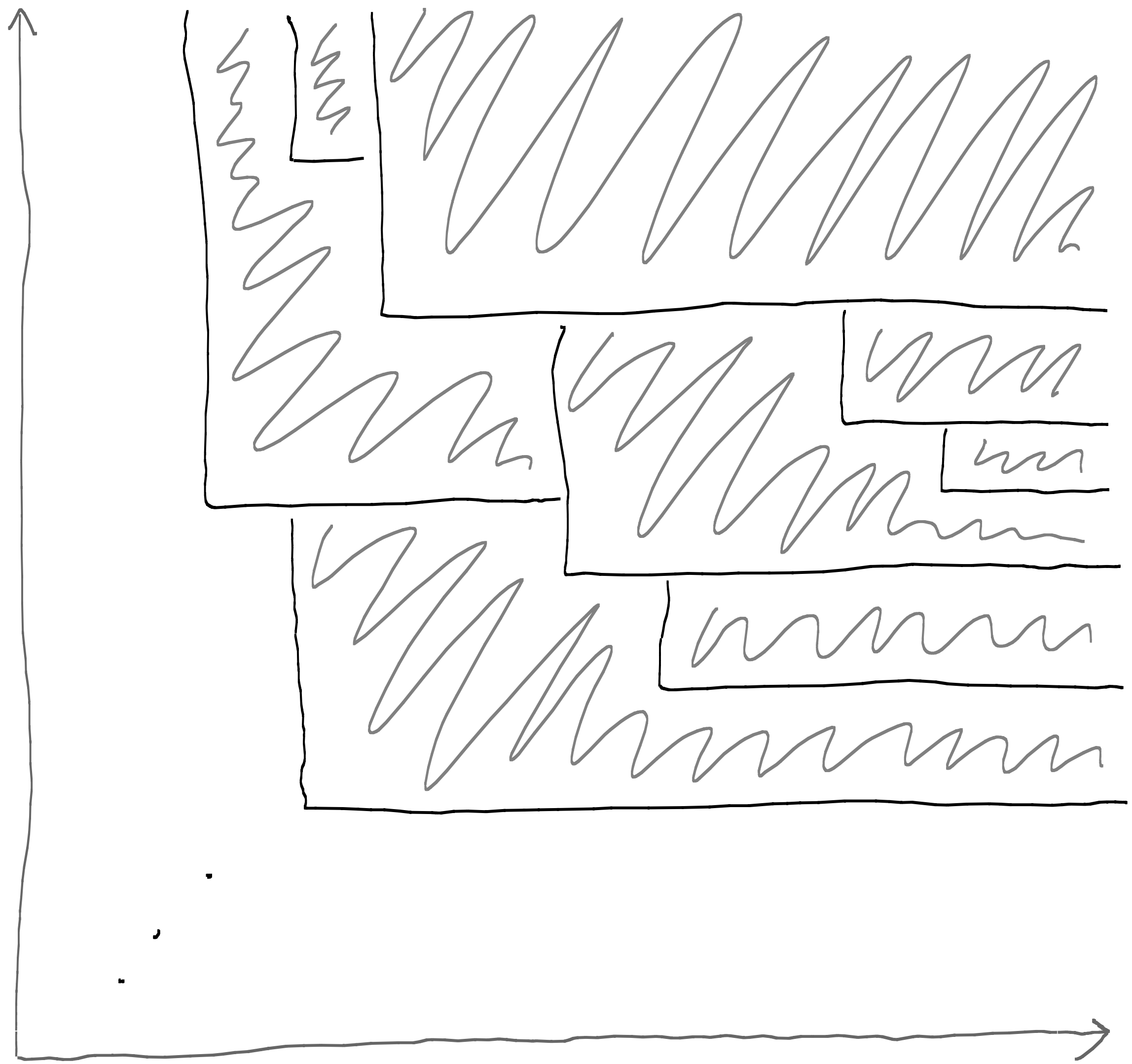


"Dickson had a sudden death trial for his prospective doctoral students: he assigned a preliminary problem which was shorter than a dissertation problem, and if the student could solve it in three months, Dickson would agree to oversee the graduate student's work. If not the student had to look elsewhere for an advisor."[3]

ω-Satisfiability

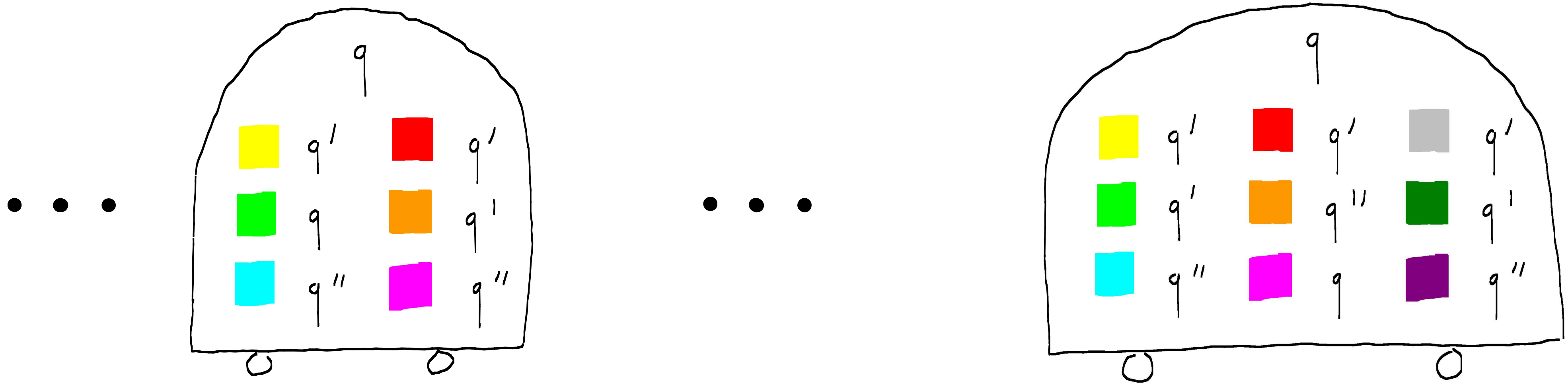
$FO_2(\sim, <, +1)$

$\downarrow$

class-memory ω-automata

$\langle Q, \Sigma, \delta, q_I, F_G, F_L \rangle$

F_G occurs inf. often and for every datum, F_L occurs inf. often



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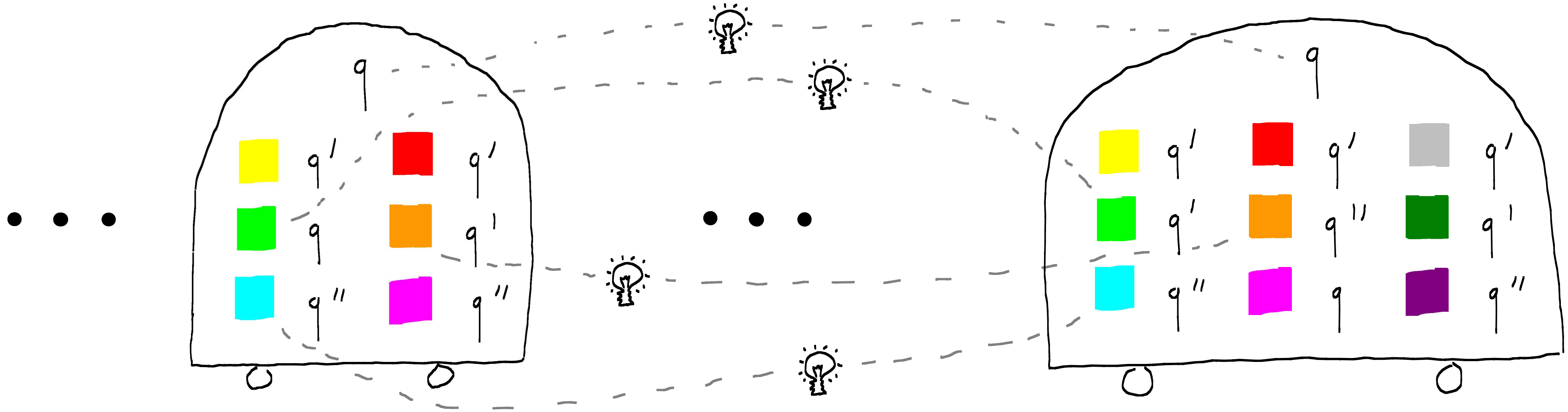

ω-Satisfiability

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$\downarrow$

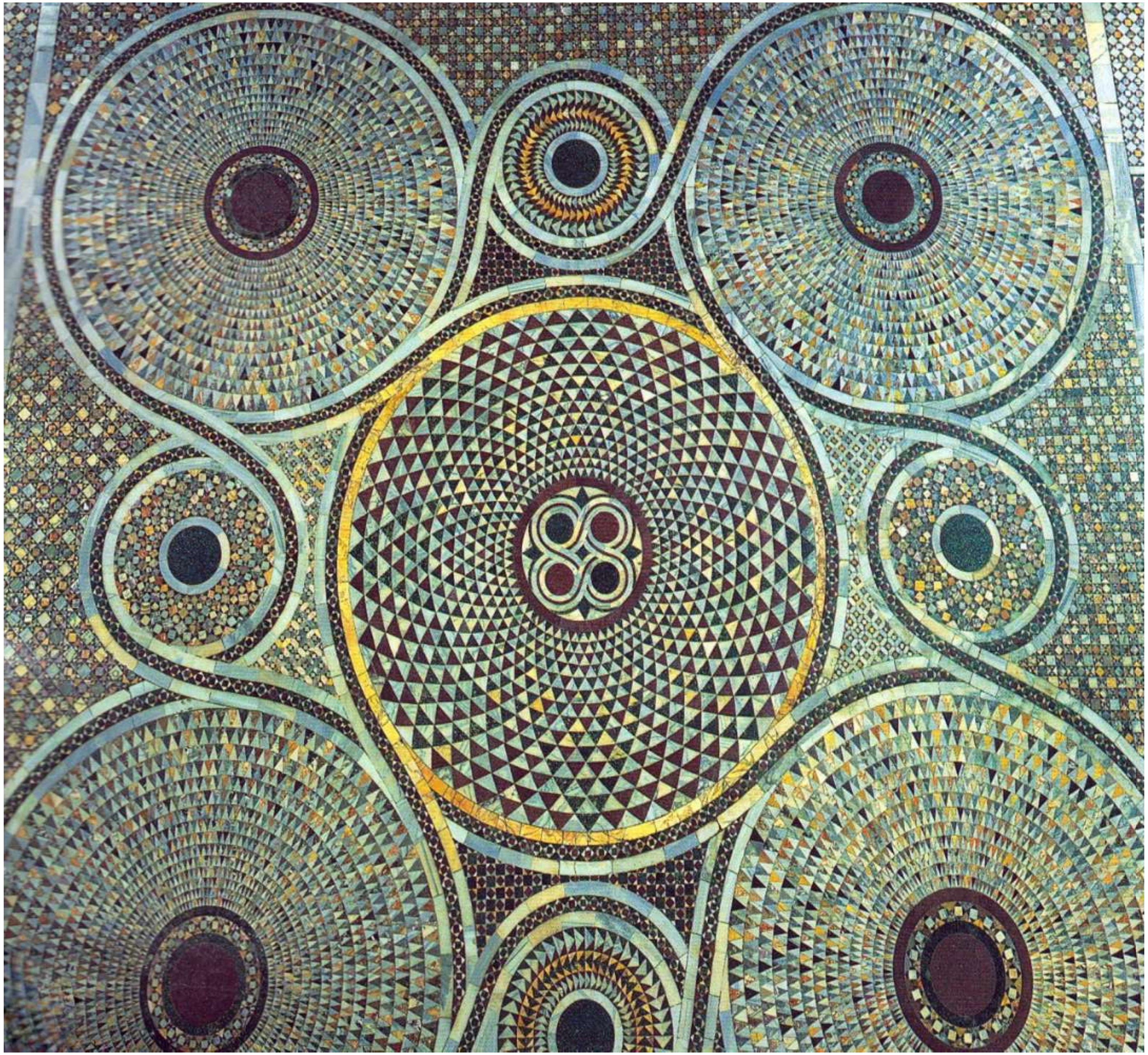
class-memory ω-automata

$\langle Q, \Sigma, \delta, q_I, F_G, F_L \rangle$

F_G occurs inf. often and for every datum, F_L occurs inf. often

$\downarrow$

counter automata



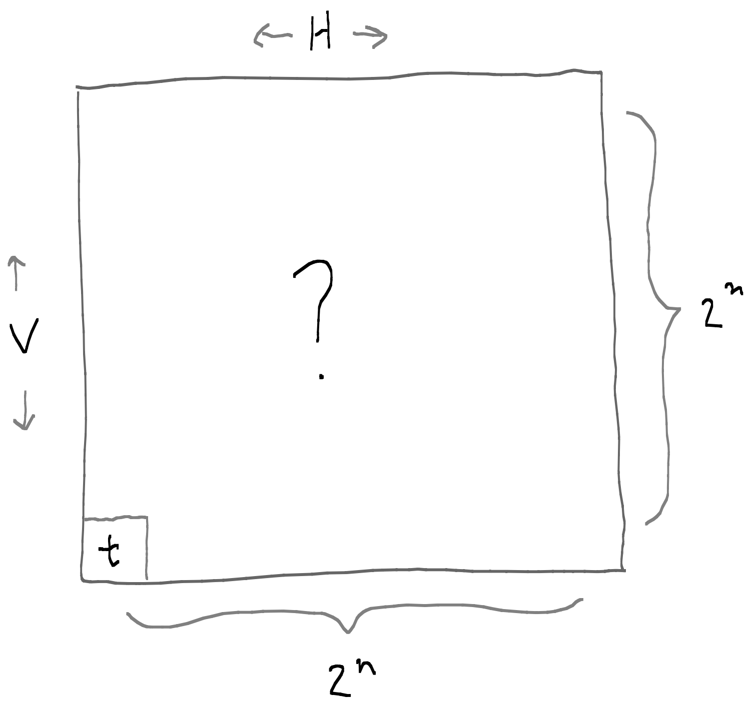
NExpTime-complete problem

[Fürer LNCS '84]

Input:

$$T, \overbrace{H, V}^{\subseteq T \times T}, t$$

Question:



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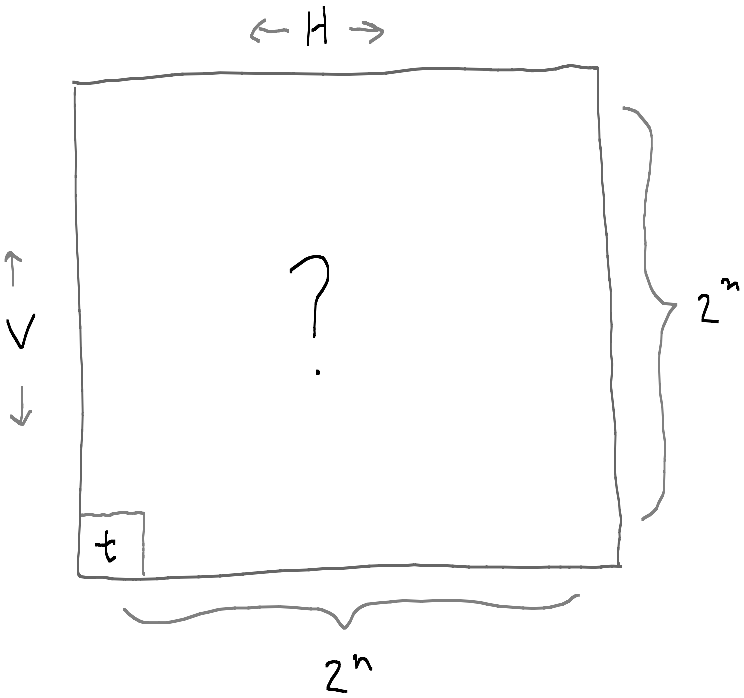

Satisfiability for $FO^2(\sim, <)$ without unary predicates is NExpTime-hard

NExpTime-complete problem

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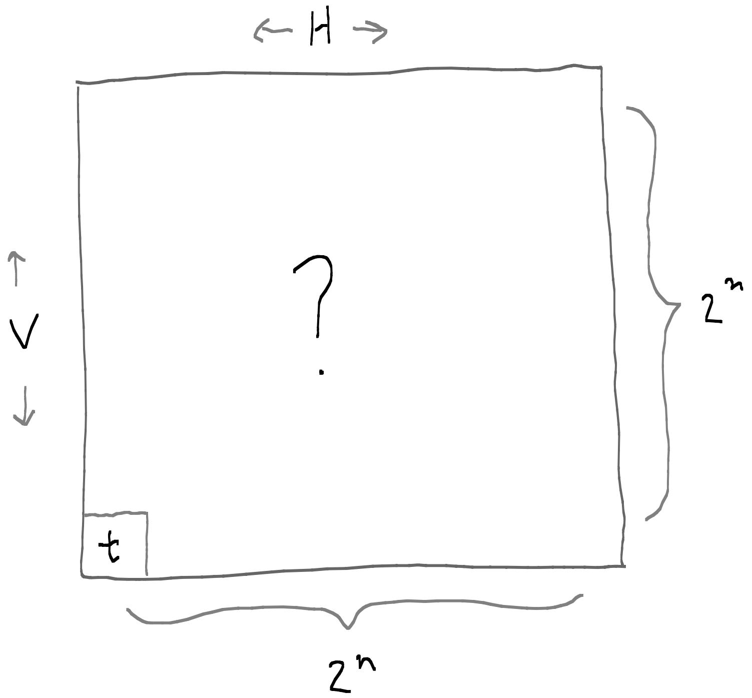
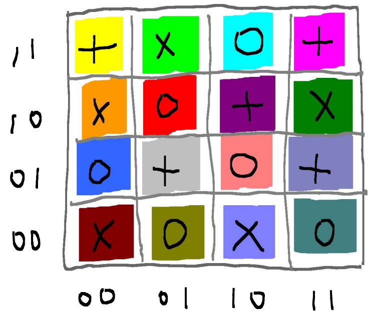


Satisfiability for $FO^2(\sim, <)$ without unary predicates is NExpTime-hard

Proof:



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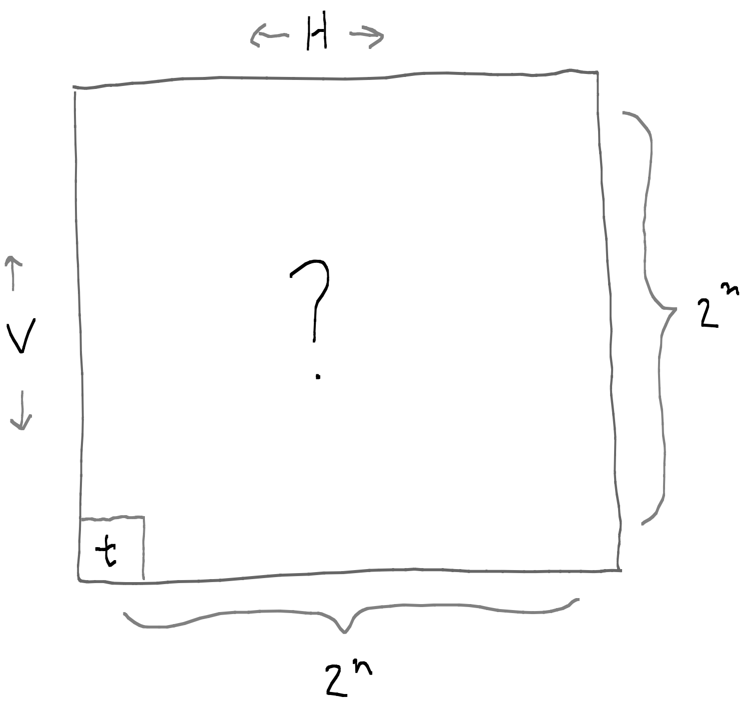

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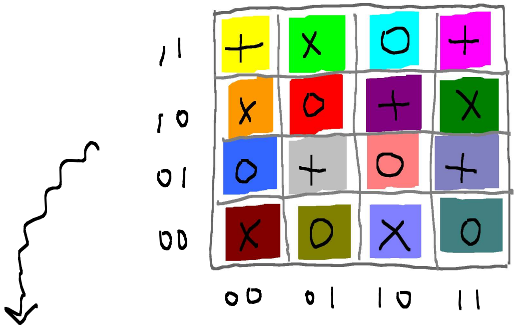
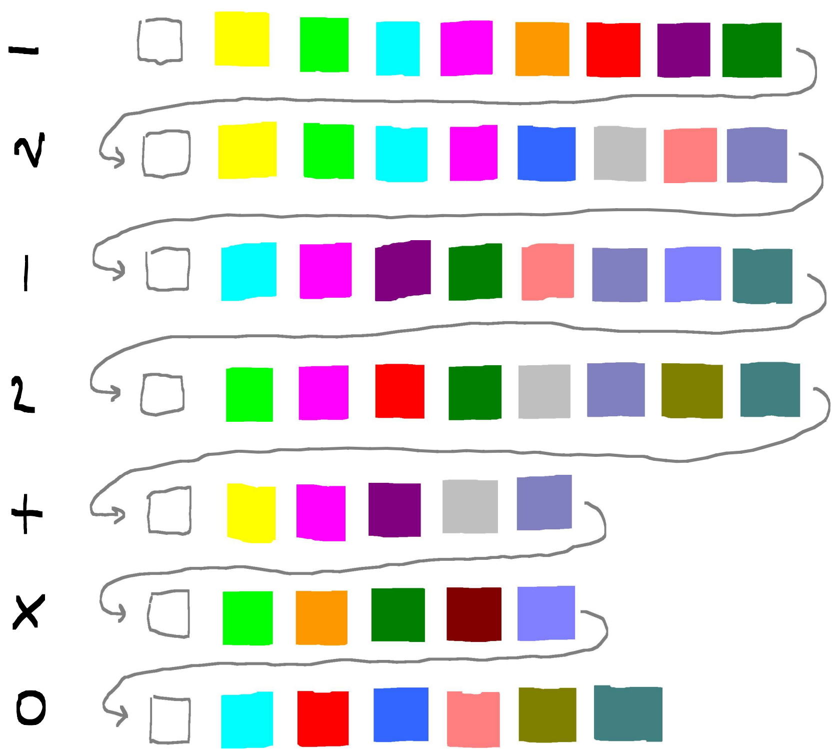




Linear temporal logic

$$G\,(r \Rightarrow \downarrow F\,(s \wedge \uparrow))$$

$$G\,(\neg r \Rightarrow \downarrow X^{-1} F^{-1} \uparrow)$$

$$G\,(s \Rightarrow \downarrow X\,(r \;U\; (t \wedge \uparrow)))$$

r r s t s r t

Satisfiability decidable?

$LTL_1^{\downarrow}(X, U)$

$LTL_2^{\downarrow}(F)$ $\qquad$ $LTL_1^{\downarrow}(F, F^{-1})$

no $\qquad$ no

[Figueira & Segoufin MFCS '09]

Satisfiability decidable?

$\mathrm{LTL}_1^{\downarrow}(X, U)$ $\xrightarrow{\text{log. space}}$ 1-way alternating 1-register automata

$\mathrm{LTL}_2^{\downarrow}(F)$

no

$\mathrm{LTL}_1^{\downarrow}(F, F^{-1})$

no

[Figueira & Segoufin MFCS '09]

Satisfiability decidable?

$LTL^{\downarrow}_{1}(X, U) \xrightarrow{\text{log. space}}$ 1-way alternating 1-register automata

cf.
[Kaminski & Francez TCS '94]
[Neven, Schwentick & Vianu ToCL '04]

$LTL^{\downarrow}_{2}(F)$

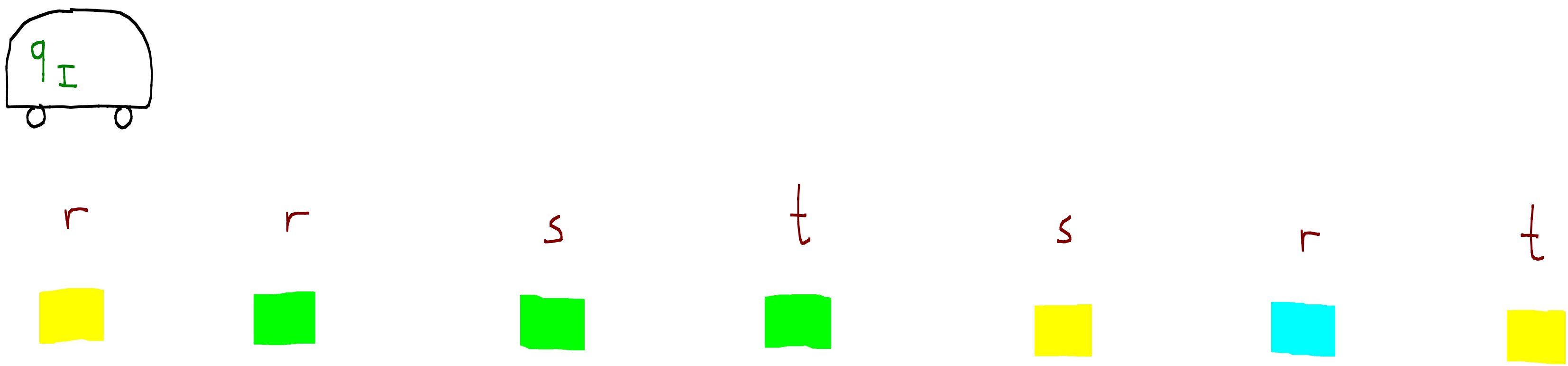
no

$LTL^{\downarrow}_{1}(F, F^{-1})$

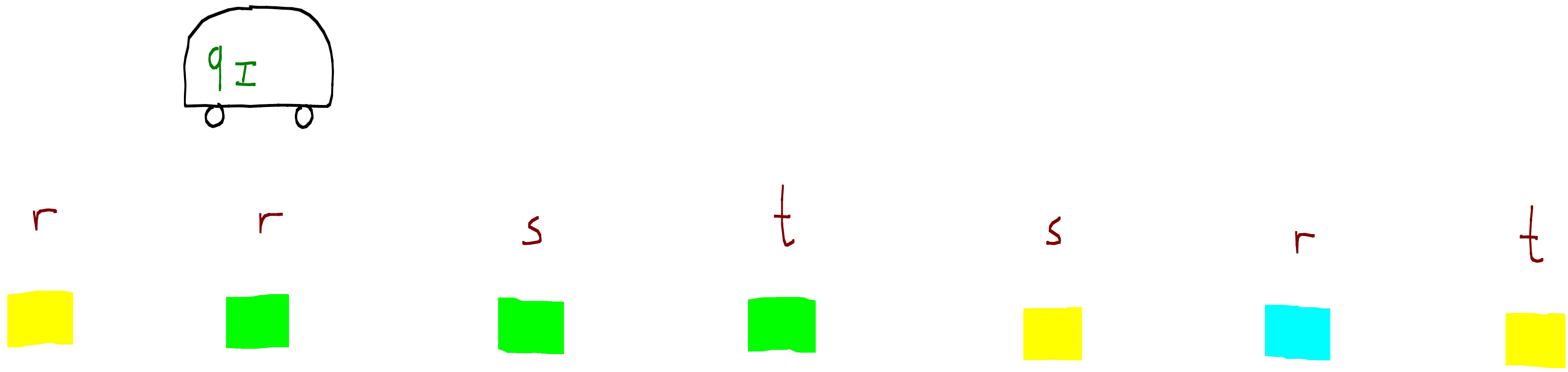
no

[Figueira & Segoufin MFCS '09]

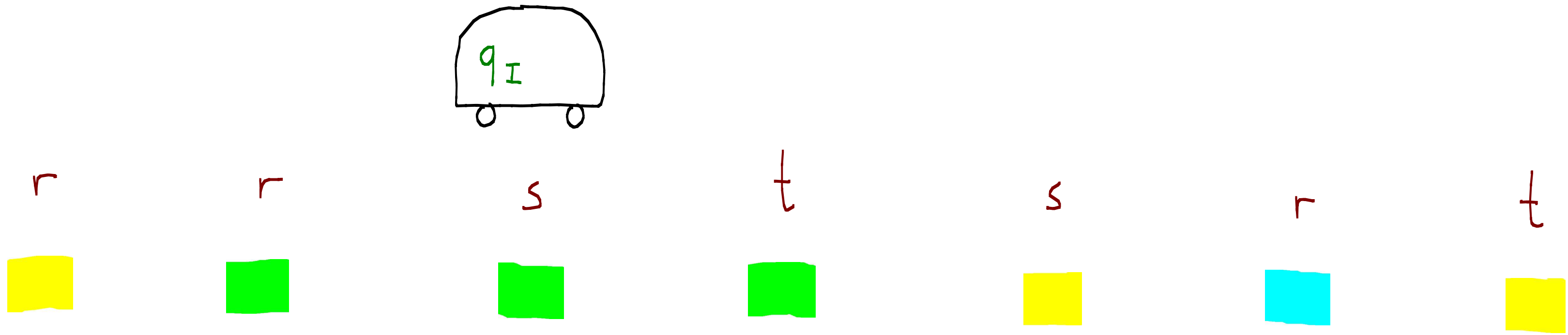
$$G\,(s \Rightarrow \downarrow X\,(r \;U\; (t \wedge \uparrow)))$$



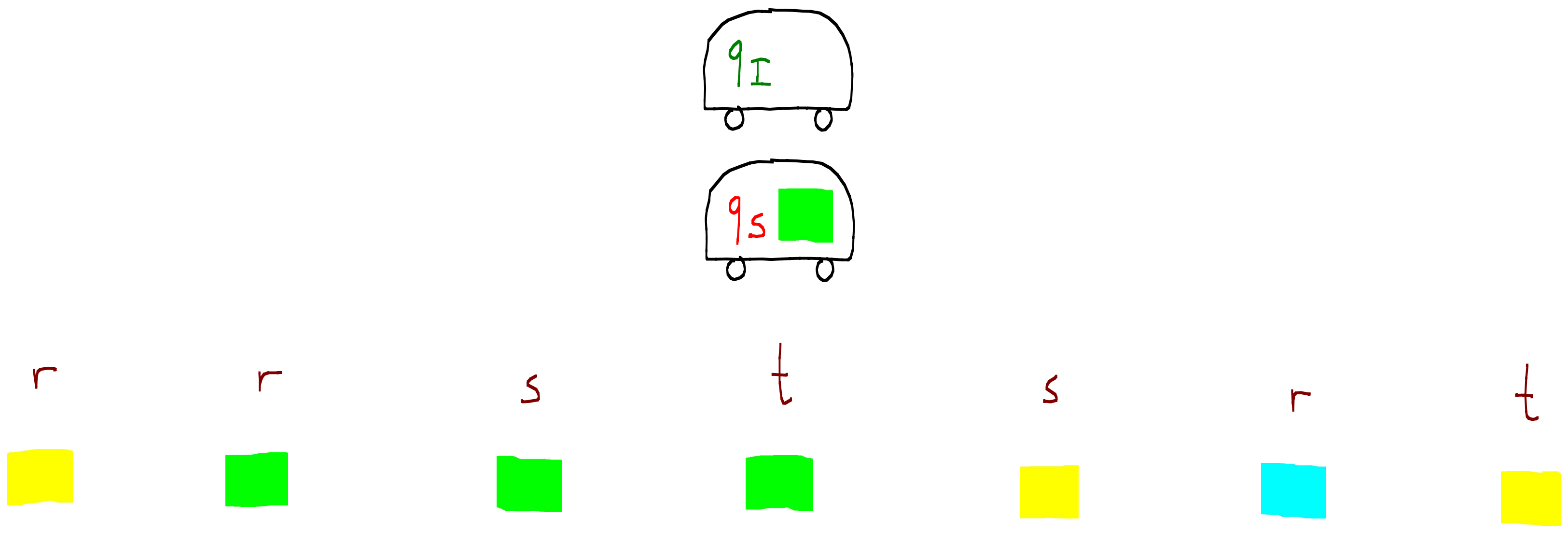
$$G\,(s \Rightarrow {\downarrow} X\,(r \;U\; (t \wedge {\uparrow})))$$



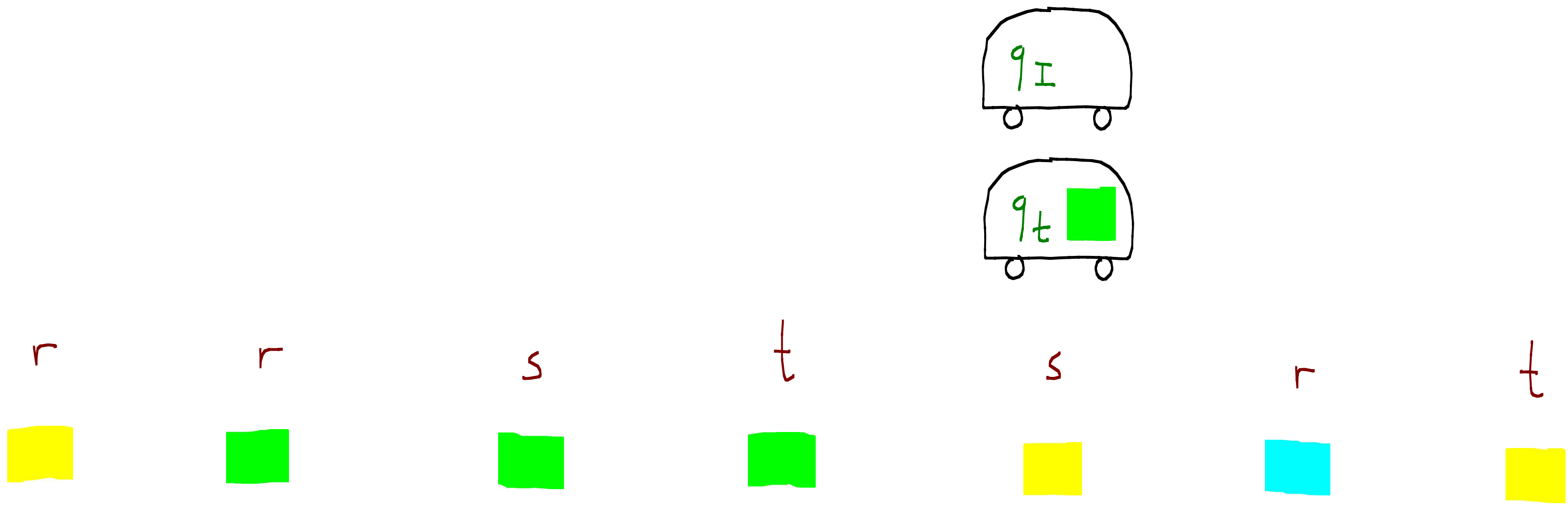
$$G\,(s \Rightarrow {\downarrow} X\,(r\,U\,(t \wedge {\uparrow})))$$



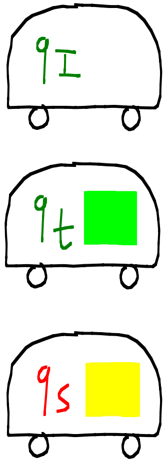
$$G\,(s \Rightarrow \downarrow X\,(r\ U\ (t \wedge \uparrow)))$$



$$G\,(s \Rightarrow {\downarrow} X\,(r\ U\ (t \wedge {\uparrow})))$$

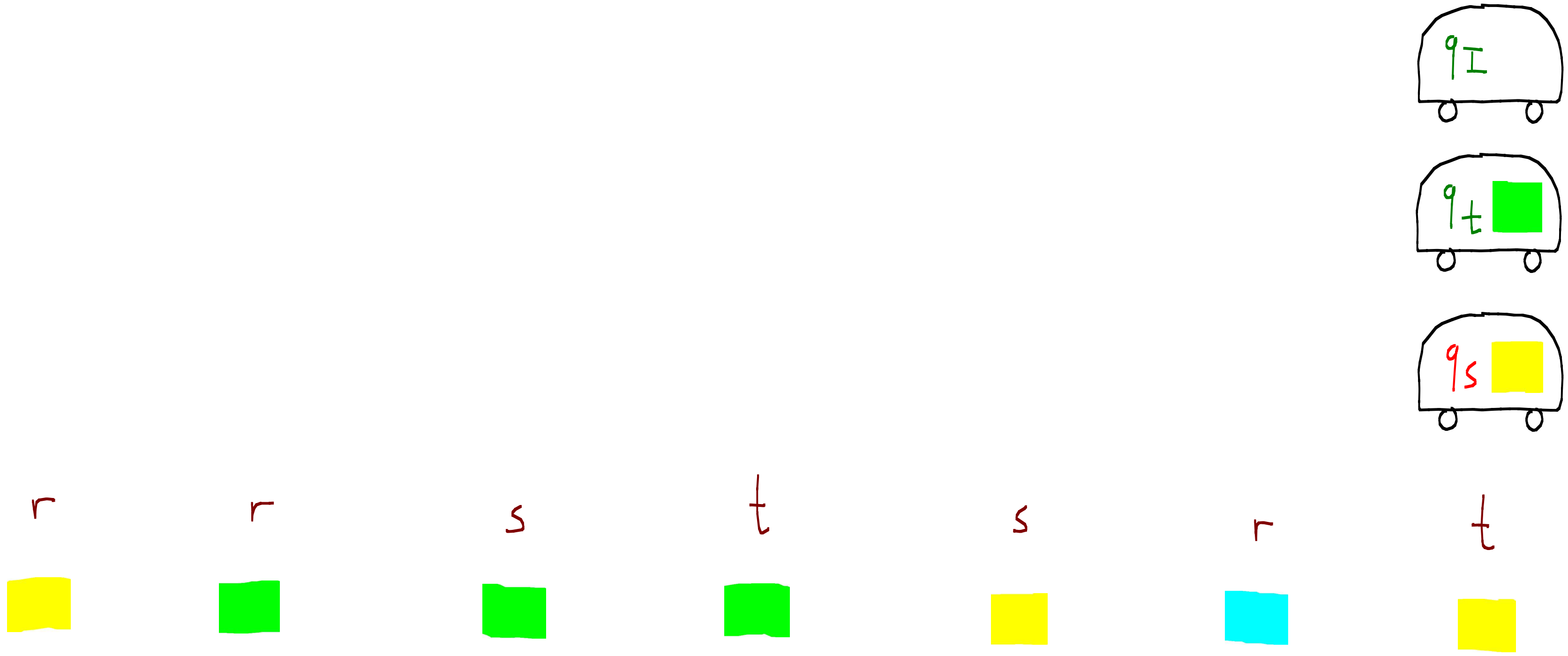


$$G\,(s \Rightarrow {\downarrow} X\,(r\ U\ (t \wedge {\uparrow})))$$

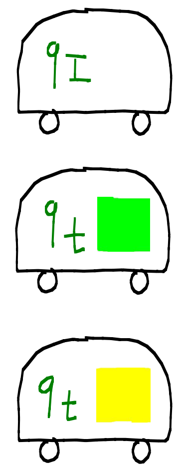


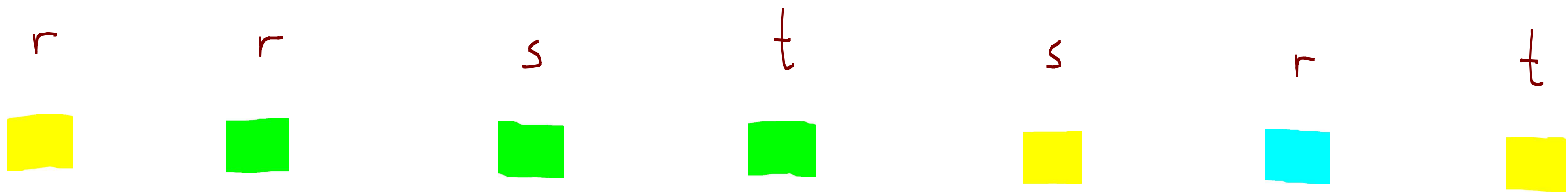
r r s t s r t

$$G\,(s \Rightarrow \downarrow X\,(r \;U\; (t \wedge \uparrow)))$$



$$G\,(s \Rightarrow \downarrow X\,(r\ U\ (t \wedge \uparrow)))$$



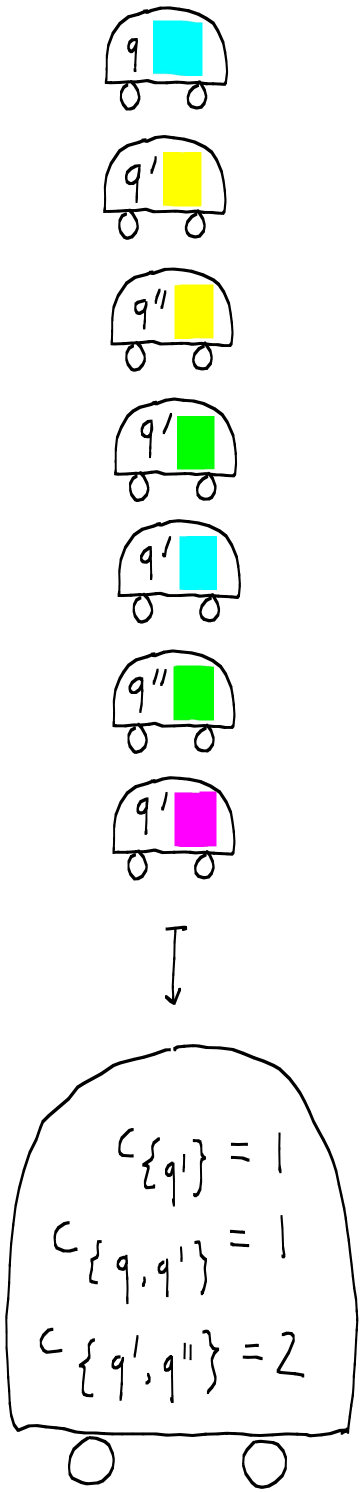


$\mathrm{LTL}_1^{\downarrow}(\mathsf{X}, \mathsf{U}) \xrightarrow{\text{log. space}}$ 1-way alternating 1-register automata

poly. space

Minsky automata

[Demri & L. ToCL '09]



$$\mathrm{LTL}^{\downarrow}_{1}(\mathsf{X},\mathsf{U}) \xrightarrow{\text{log. space}} \text{1-way alternating 1-register automata}$$

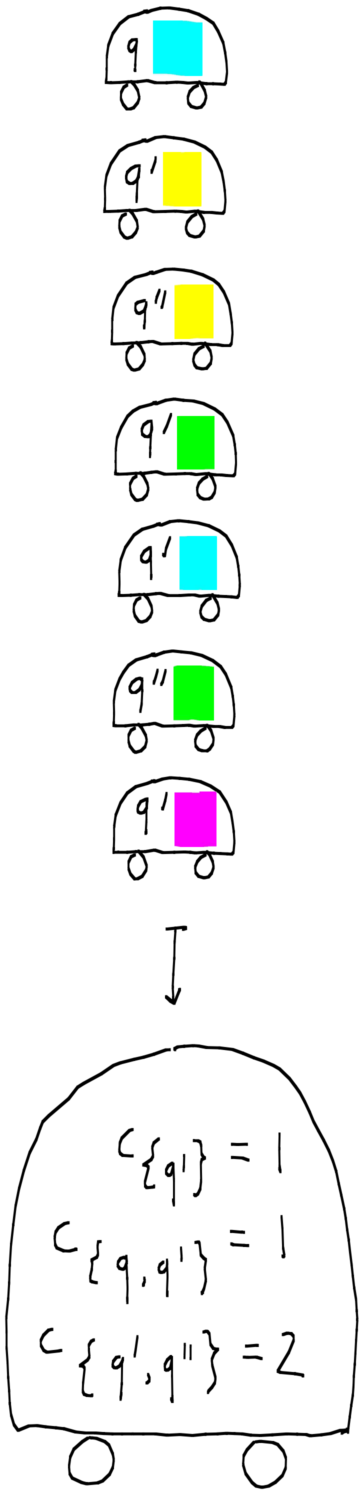
[Demri & L. ToCL '09]

$$\xrightarrow{\text{poly. space}}$$

gainy Minsky automata

Nonemptiness :

in ACKERMANN [Figueira et al. LICS '11]



$\mathrm{LTL}^{\downarrow}_{1}(X, U)$ $\xrightarrow{\text{log. space}}$ 1-way alternating 1-register automata

[Demri & L. ToCL '09]

poly. space

log. space

gainy

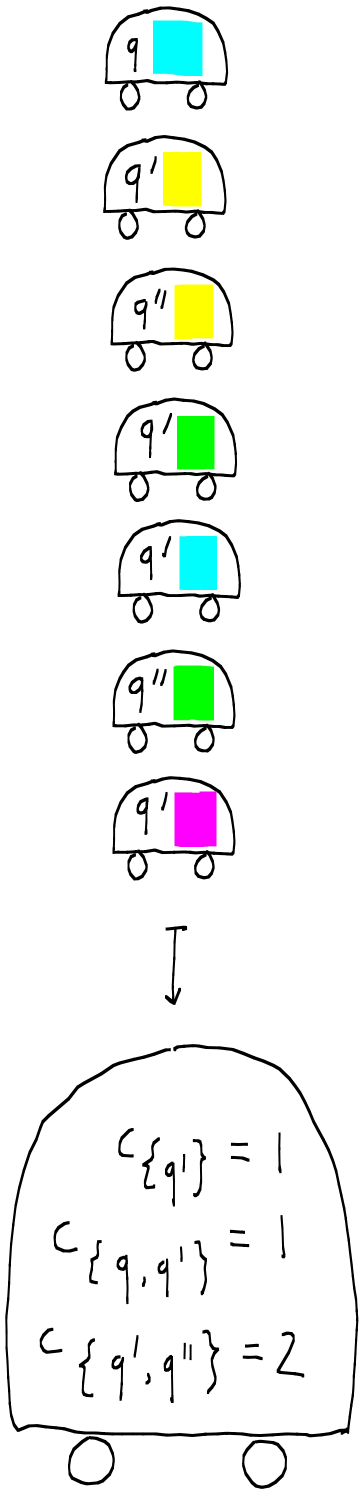
Minsky automata

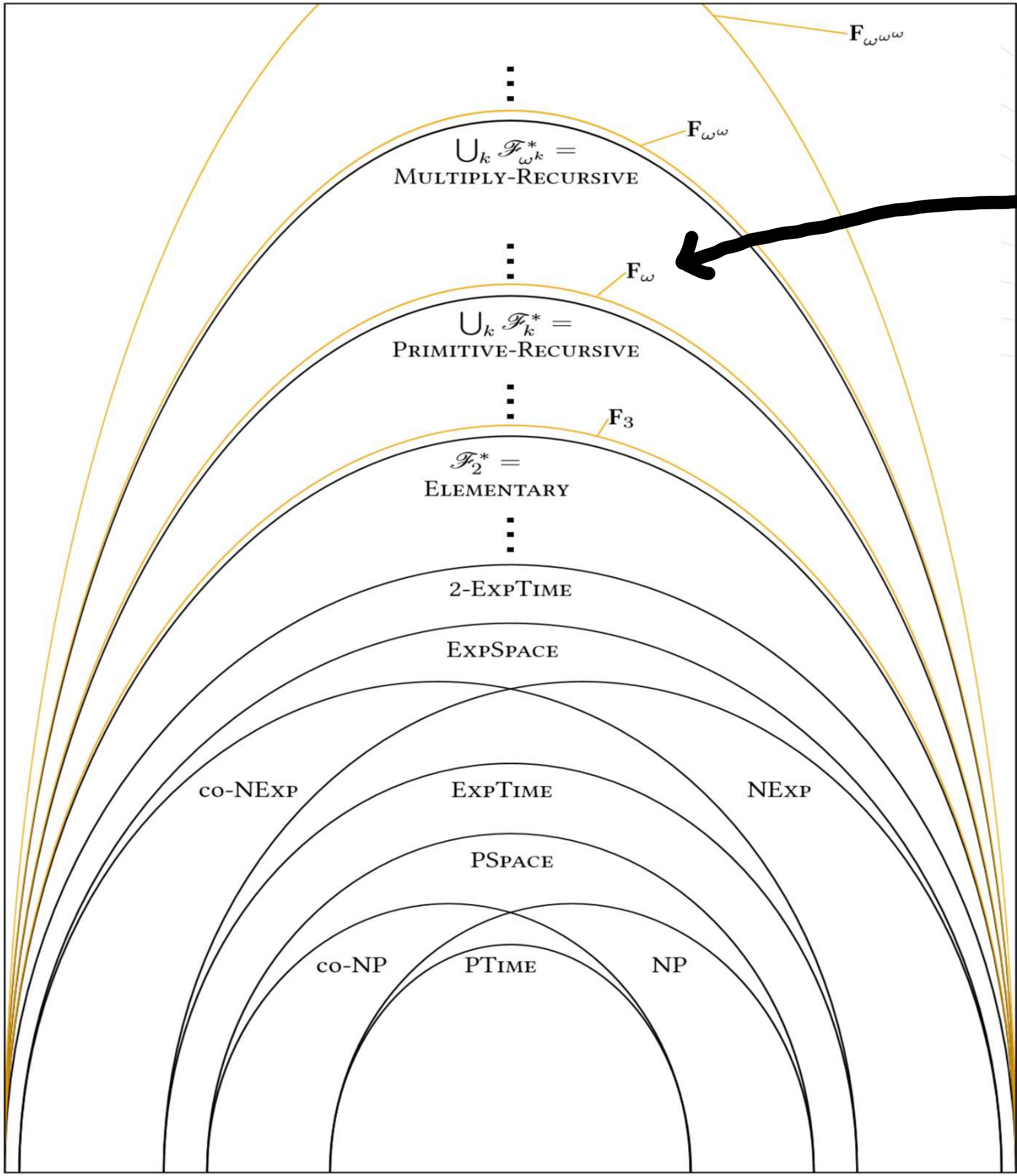
Nonemptiness :

in ACKERMANN [Figueira et al. LICS '11]

ACKERMANN-hard [Schnoebelen IPL '02, MFCS '10]

[Schmitz ToCT '16]




$\mathbf{F}_{\omega^{\omega^{\omega}}}$
$\mathbf{F}_{\omega^{\omega}}$
$\bigcup_k \mathscr{F}^*_{\omega^k} =$
Multiply-Recursive
$\mathbf{F}_{\omega}$
$\bigcup_k \mathscr{F}^*_k =$
Primitive-Recursive
$\mathbf{F}_3$
$\mathscr{F}^*_2 =$
Elementary
2-ExpTime
ExpSpace
co-NExp
ExpTime
NExp
PSpace
co-NP
PTime
NP

Values of *A*(*m*, *n*)

*m**n*	**0**	**1**	**2**	**3**	**4**	***n***
0	1	2	3	4	5	$n+1$
1	2	3	4	5	6	$n+2=2+(n+3)-3$
2	3	5	7	9	11	$2n+3=2\cdot(n+3)-3$
3	5	13	29	61	125	$2^{(n+3)}-3$
4	13 $=2^{2^{2}}-3$	65533 $=2^{2^{2^{2}}}-3$	$2^{65536}-3$ $=2^{2^{2^{2^{2}}}}-3$	$2^{2^{65536}}-3$ $=2^{2^{2^{2^{2^{2}}}}}-3$	$2^{2^{2^{65536}}}-3$ $=2^{2^{2^{2^{2^{2^{2}}}}}}-3$	$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n+3}-3$
5	65533 $=2\uparrow\uparrow\uparrow 3-3$	$2\uparrow\uparrow\uparrow 4-3$	$2\uparrow\uparrow\uparrow 5-3$	$2\uparrow\uparrow\uparrow 6-3$	$2\uparrow\uparrow\uparrow 7-3$	$2\uparrow\uparrow\uparrow (n+3)-3$
6	$2\uparrow\uparrow\uparrow\uparrow 3-3$	$2\uparrow\uparrow\uparrow\uparrow 4-3$	$2\uparrow\uparrow\uparrow\uparrow 5-3$	$2\uparrow\uparrow\uparrow\uparrow 6-3$	$2\uparrow\uparrow\uparrow\uparrow 7-3$	$2\uparrow\uparrow\uparrow\uparrow (n+3)-3$

counter automata

$\downarrow$ log. space [Bojańczyk et al. ToCL'11]

$FO^2(\sim, <, +1)$

$$q_t \xrightarrow[inc(c_r)]{r} q_r \xrightarrow[inc(c_r)]{r} q_r \xrightarrow[dec(c_r)]{s} q'_s \xrightarrow[inc(c_s)]{\varepsilon} q_s \ \cdots$$

$$\bigwedge_{\substack{q,q',q'',q'''\in Q \\ a,a'\in \Sigma\cup\{\varepsilon\} \\ c\in C}} \forall x \forall y \neg \big(\langle q,a,inc,c,q'\rangle(x) \wedge \langle q'',a',inc,c,q'''\rangle(y) \wedge x<y \wedge x\sim y \big)$$

$$\bigwedge_{\substack{q,q'\in Q \\ a\in \Sigma\cup\{\varepsilon\} \\ c\in C}} \forall x \Big(\langle q,a,dec,c,q'\rangle(x) \Rightarrow \bigvee_{\substack{q'',q'''\in Q \\ a'\in \Sigma\cup\{\varepsilon\}}} \exists y \big(\langle q'',a',inc,c,q'''\rangle(y) \wedge y<x \wedge y\sim x \big)\Big)$$

counter automata

$\big\downarrow$ log. space [Bojańczyk et al. ToCL'11]

$FO^2(\sim, <, +1)$

$$q_t \xrightarrow[\mathit{inc}(c_r)]{r} q_r \xrightarrow[\mathit{inc}(c_r)]{r} q_r \xrightarrow[\mathit{dec}(c_r)]{s} q_s' \xrightarrow[\mathit{inc}(c_s)]{\varepsilon} q_s \quad \dots$$

$$\bigwedge_{\substack{q,q',q'',q'''\in Q\\ a,a'\in\Sigma\cup\{\varepsilon\}\\ c\in C}} G\,\langle q,a,\mathit{inc},c,q'\rangle \Rightarrow {\downarrow} x\, G\,\langle q'',a',\mathit{inc},c,q'''\rangle \Rightarrow \neg{\uparrow}$$

$$\bigwedge_{\substack{q,q'\in Q\\ a\in\Sigma\cup\{\varepsilon\}\\ c\in C}} \forall x\,\Big(\langle q,a,\mathit{dec},c,q'\rangle(x) \Rightarrow \bigvee_{\substack{q'',q'''\in Q\\ a'\in\Sigma\cup\{\varepsilon\}}} \exists y\,\big(\langle q'',a',\mathit{inc},c,q'''\rangle(y) \wedge y<x \wedge y\sim x\big)\Big)$$

counter automata $\xrightarrow{\text{log. space}}$ $FO^2(\sim, <, +1)$ [Bojańczyk et al. ToCL'11]

gainy Minsky automata $\xrightarrow{\text{log. space}}$ $LTL^{\downarrow}_{1}(X, U)$ [Demri & L. ToCL'09]

$$q_t \xrightarrow[inc(c_r)]{r} q_r \xrightarrow[inc(c_r)]{r} q_r \xrightarrow[dec(c_r)]{s} q'_s \xrightarrow[inc(c_s)]{\varepsilon} q_s \quad \ldots$$

$$\bigwedge_{\substack{q,q',q'',q'''\in Q\\ a,a'\in\Sigma\cup\{\varepsilon\}\\ c\in C}} G\,\langle q,a,inc,c,q'\rangle \Rightarrow \downarrow XG\,\langle q'',a',inc,c,q'''\rangle \Rightarrow \neg\uparrow$$

$$\bigwedge_{\substack{q,q',q'',q'''\in Q\\ a,a'\in\Sigma\cup\{\varepsilon\}\\ c\in C}} G\,\langle q,a,inc,c,q'\rangle \Rightarrow \downarrow X\left(\neg\langle q'',a',is0,c,q'''\rangle \; U \bigvee_{\substack{q^{IV},q^{V}\in Q\\ a''\in\Sigma\cup\{\varepsilon\}}} \langle q^{IV},a'',dec,c,q^{V}\rangle \wedge \uparrow\right)$$

Corollary

$$\left\{ f(\mathrm{str}(L(\varphi))) \;:\; \begin{array}{l} \varphi \text{ in } LTL_1^{\downarrow}(X, U) \\ f \text{ homomorphism} \end{array} \right\}$$

$$\|$$

$$\left\{ f(\mathrm{str}(L(A))) \;:\; \begin{array}{l} A \text{ a } IARA_1 \\ f \text{ homomorphism} \end{array} \right\}$$

$$\|$$

$$\left\{ L(\mathcal{C}) \;:\; \mathcal{C} \text{ a } GMA \right\}$$

Closure properties of $IARA_1$

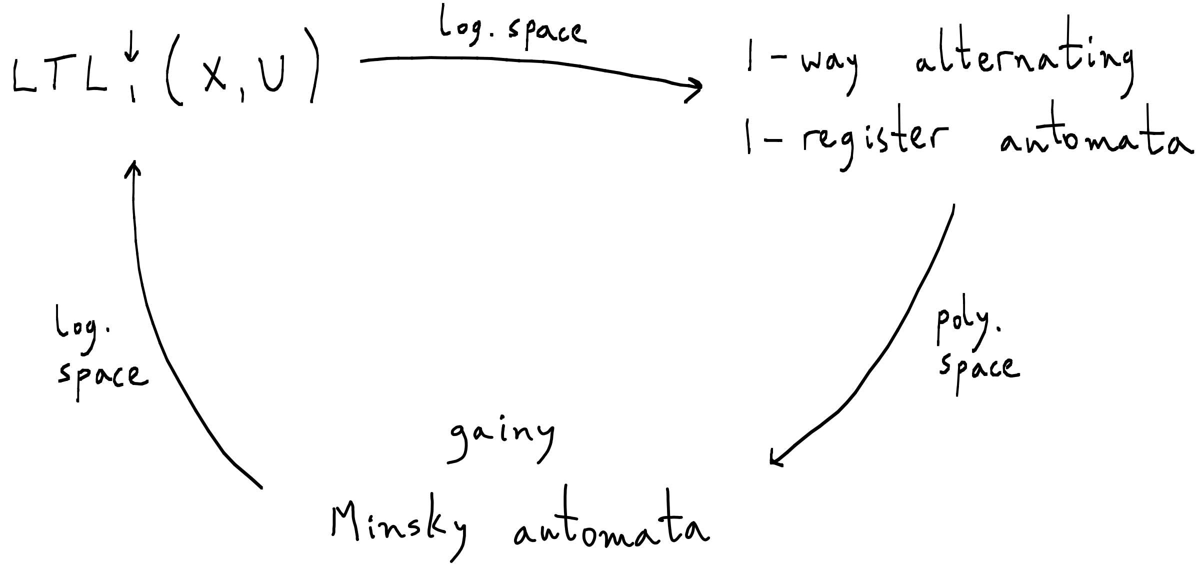
intersection ✓

union ✓

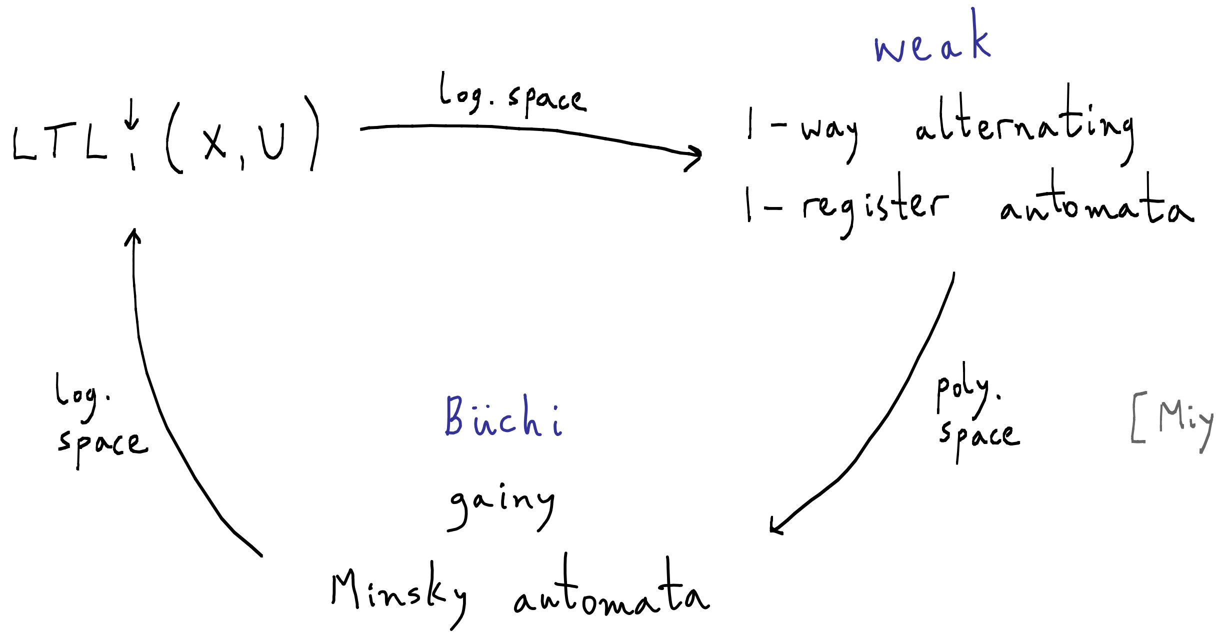
complement ✓

renaming ✗

ω-Satisfiability



ω-Satisfiability



[Miyano & Hayashi TCS '84]

ω-Satisfiability

$$\begin{tikzcd} \mathrm{LTL}_1^{\downarrow}(X, U) \arrow[r, "\text{log. space}"] & \text{weak 1-way alternating 1-register automata} \arrow[dl, "\text{poly. space}"] \\ \text{Büchi gainy Minsky automata} \arrow[u, "\text{log. space}"] & \end{tikzcd}$$

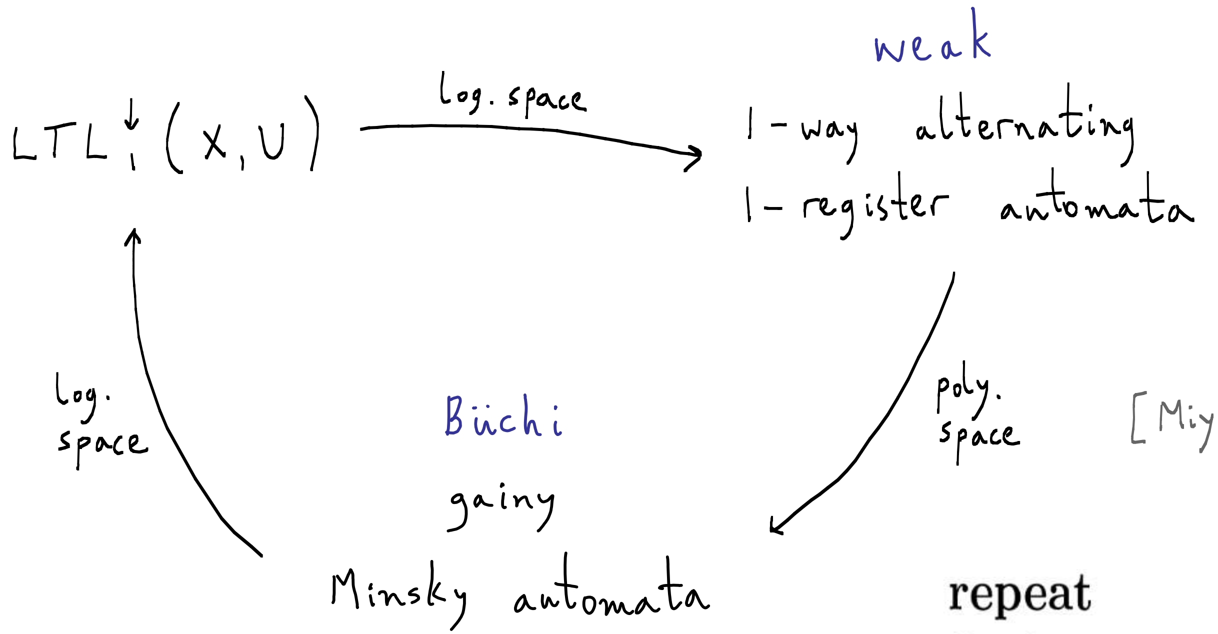
[Miyano & Hayashi TCS '84]

Nonemptiness Π_1^0-complete!

[Ouaknine & Worrell FoSSaCS '06]

cf. [Mayr TCS '03]

ω-Satisfiability



[Miyano & Hayashi TCS '84]

Nonemptiness Π_1^0-complete!

[Ouaknine & Worrell FoSSaCS '06]

cf. [Mayr TCS '03]

repeat
{ $D' := D$;
  while $D' > 0$
  { simulate $\mathcal{C}$ using C_1 and C_2 as follows:
   - if $\mathcal{C}$ accepts, $\widehat{\mathcal{C}}$ stops
   - whenever C_1 or C_2 is decremented, increment D'
   - whenever C_1 or C_2 is incremented, decrement D'
   - after each step of $\mathcal{C}$, increment C' and decrement D'
   - if $D' = 0$, exit the simulation;
   $D' = D' + C_1 + C_2 + C' - 1$; $C_1, C_2, C' := 0$ };
  $D := D + 1$ }

Safety $LTL^{\downarrow}_{1}(X, U)$

no positive occurrences of U

$$G\,(s \Rightarrow \downarrow X\,(r\;U\;(t \wedge \uparrow)))$$

$$G\,(s \Rightarrow \downarrow X \neg(\neg\uparrow\;U\;(\neg t \wedge \uparrow)))$$

Safety $\mathrm{LTL}_1^{\downarrow}(\mathrm{X}, \mathrm{U})$

no positive occurrences of U

$$\mathrm{G}(s \Rightarrow {\downarrow} \mathrm{X}(r \,\mathrm{U}\, (t \wedge \uparrow)))$$

$$\mathrm{G}(s \Rightarrow {\downarrow} \mathrm{X} \neg (\neg \uparrow \mathrm{U} (\neg t \wedge \uparrow)))$$

ω-Satisfiability is ExpSpace-complete!

[L. ToCL '11]

cf. [Rackoff TCS '78]

[Bouyer, Markey, Ouaknine, Schnoebelen & Worrell FACJ '12]