A logical approach to Isomorphism Testing and Constraint Satisfaction

Oleg Verbitsky

Humboldt University of Berlin, Germany

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Part 3: $\mathrm{FO}^2_{\#}$-definable graphs.

Outline

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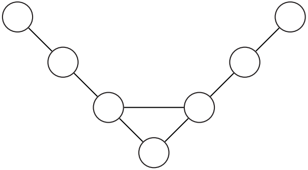
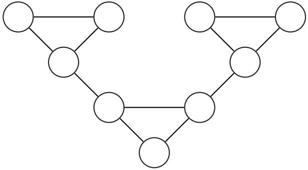
- Every tree; because every tree is characterizable by Color Refinement [Edmonds 65] (a proof is postponed).

Example

Which one of the following two graphs is definable in $\mathrm{FO}^2_{\#}$?





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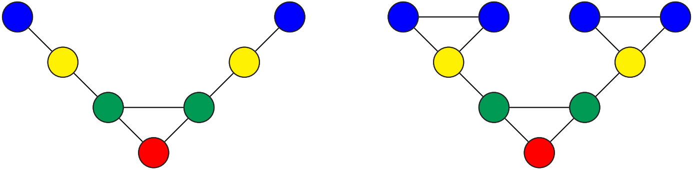

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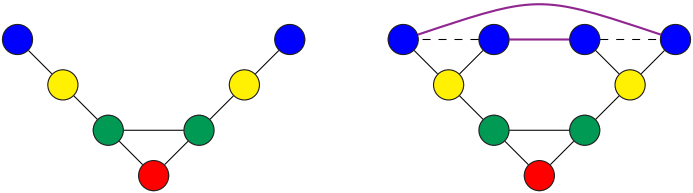

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Local structure of definable graphs

Definition

Call the stable color classes of vertices cells.

Let X be a cell in G.

- $G[X]$ is regular.
- If G is definable, then $G[X]$ must be a unigraph (to be proved shortly).

Complete list of regular unigraphs [Johnson 1975]

- Complete and empty graphs. (homogeneous)
- Matchings mK_2 and their complements. (heterogeneous)
- The cycle C_5. (heterogeneous)

Local structure of definable graphs

Let X, Y be cells in G.

- $G[X, Y]$ is bi-regular.
- If G is definable, then $G[X, Y]$ is identified by its degrees.

Complete list of such bi-regular graphs [Koren 1976]

- Complete bipartite graphs and empty graphs. (isotropic)
- Forests of stars $sK_{1,t}$, $s \geq 2$, and their bipartite complements. (anisotropic)

Local structure of definable graphs

Lemma

If G is definable, then for any cells X, Y

(A) *$G[X]$ is from the Johnson list.*

(B) *$G[X, Y]$ is from the Koren list.*

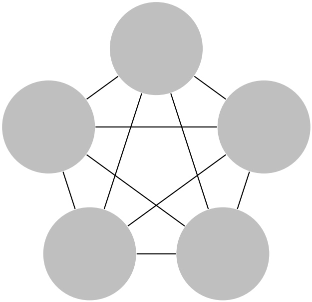
Proof of (A)

Assume that $G[X]$ is not a unigraph, that is, there is a regular graph $F \not\cong G[X]$ of the same degree with as many vertices. Change G on X so that $G'[X] \cong F$, where G' denotes the modified graph. Then

- For every vertex v, $C^i(v)$ is the same in G and G' (induction on i);
- Therefore, CR does not distinguish G and G';
- $G \not\cong G'$ because any isomorphism respects the color classes and should be an isomorphism also between $G[X]$ and $G'[X]$.

Outline

Cell-Graph



– *a cell*

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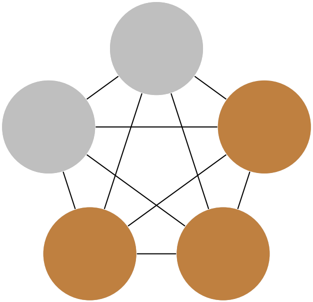

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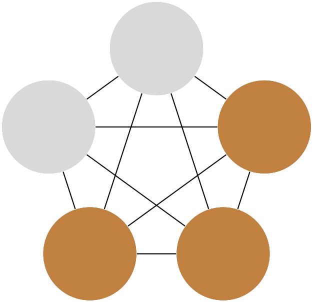

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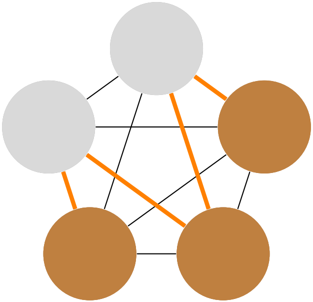

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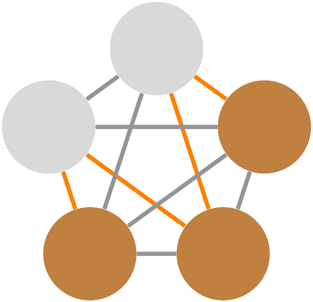

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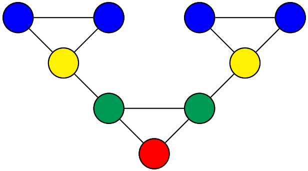
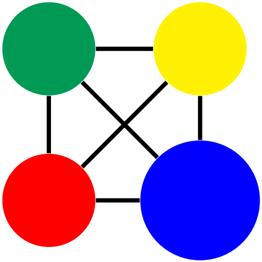
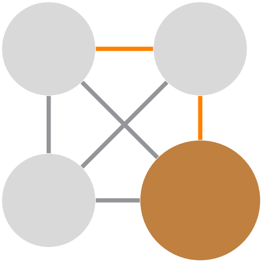
- Isotropic edges (empty/complete bipartite)

Example







Global structure of definable graphs

Configurations forbidden in the cell-graph of a definable graph:

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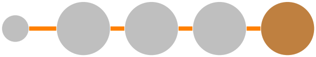

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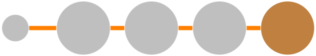
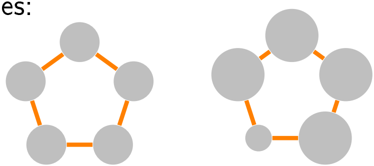
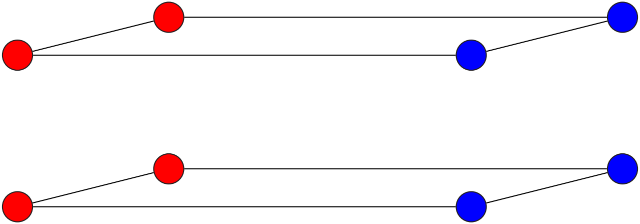


(F) Anisotropic cycles:



Proof-sketch of Condition (C)



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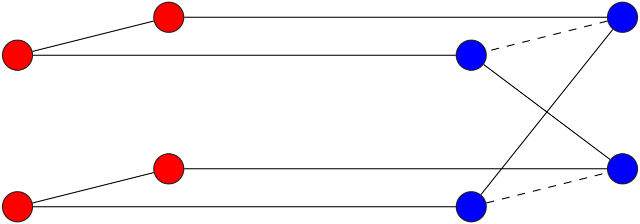

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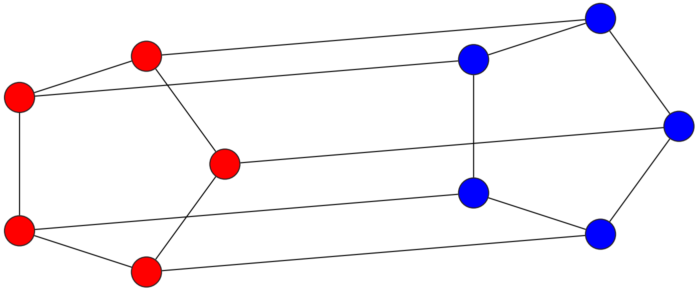

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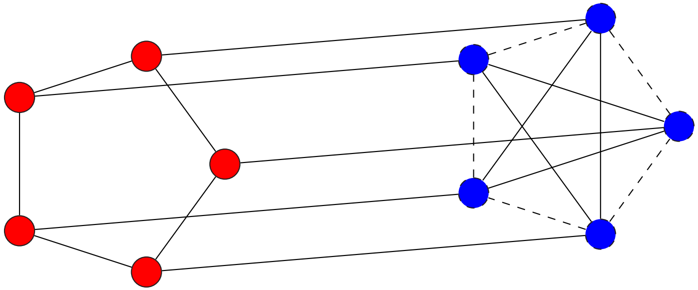

Characterization of definable graphs

Theorem (Arvind, Köbler, Rattan, V. 2015 and Kiefer, Schweitzer, Selman 2015)

Conditions (A)–(F) are both necessary and sufficient for definability of a graph in $\mathrm{FO}^2_{\#}$.

Corollary

The class of graphs definable in $\mathrm{FO}^2_{\#}$ *is recognizable in polynomial time.*

Exercise

Using the characterization of $\mathrm{FO}^2_{\#}$-definable graphs, prove that every forest is definable in $\mathrm{FO}^2_{\#}$.

Outline

References

- V. Arvind, J. Köbler, G. Rattan, and O. Verbitsky. On the power of color refinement. FCT'15.
- S. Kiefer, P. Schweitzer, and E. Selman. Graphs identified by logics with counting. MFCS'15.