A logical approach to Isomorphism Testing and Constraint Satisfaction

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Part 4: $\text{FO}_2^\#$ and linear-programming techniques.
1. The graph canonization problem

2. Fractional isomorphism and compactness

3. FO\textsuperscript{2}#-definable graphs are compact

4. Tinhofer’s canonization algorithm

5. References
Outline

1. The graph canonization problem

2. Fractional isomorphism and compactness

3. $\text{FO}^2_\#$-definable graphs are compact

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Canonizing almost all graphs

- All graphs on \( n \) vertices
- Efficiently canonizable graphs

Discrete graphs

Almost all graphs
Canonizing almost all graphs

All graphs on $n$ vertices
Efficiently canonizable graphs
Graphs definable in $\text{FO}_\#^2$
Discrete graphs
Almost all graphs
Canonizing almost all graphs

All graphs on $n$ vertices
Efficiently canonizable graphs
Compact graphs
Graphs definable in $\text{FO}_2^\#$
Discrete graphs
Almost all graphs
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1. The graph canonization problem
2. Fractional isomorphism and compactness
3. $\text{FO}_\#^2$-definable graphs are compact
4. Tinhofer’s canonization algorithm
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Consider graphs $G$ and $H$ with adjacency matrices $A$ and $B$ resp. $G \cong H$ iff there is a permutation matrix $X$ such that

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$$AX = XB. \quad (1)$$

$X = (x_{ij})$ is doubly stochastic (d.s.) if $x_{ij} \geq 0$, $\sum_i x_{ij} = 1$ for all $j$, and $\sum_j x_{ij} = 1$ for all $i$.

**Definition**

$G$ and $H$ are fractionally isomorphic if (1) is true for some d.s. $X$. 
Fractional isomorphism and equivalent concepts

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**Theorem [Ramana, Scheinerman, Ullman 94; Immerman, Lander 90]**

The following three conditions are equivalent:

- $G$ and $H$ are fractionally isomorphic,
- $G$ and $H$ are indistinguishable by Color Refinement,
- $G$ and $H$ are indistinguishable in $\text{FO}^2\#$.
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Let $S(G) = \{ X - \text{d.s.} : AX =XA \}$, the set of all fractional automorphisms of $G$.

- This is a polytope in $\mathbb{R}^{n^2}$.
- Automorphisms of $G$ (permutation matrices) are extreme points of $S(G)$. 

If $G \sim H$, then all extreme points of $S(G,H)$ are integral; if $G \not\sim H$, then $S(G,H)$ has no integral point.
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$G$ is **compact** if $S(G)$ has no other extreme points.
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Equivalently,

- all extreme points of $S(G)$ are integral, or
- every fractional automorphism of $G$ is a convex combination of automorphisms of $G$. 
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If $G$ is known to be compact, then $G \cong H$ can be tested by computing an extreme point of $S(G, H) = \{ X - \text{d.s.} : AX =XB \}$:

- If $G \cong H$, then all extreme points of $S(G, H)$ are integral;
- If $G \not\cong H$, then $S(G, H)$ has no integral point.
Basic facts: Complete graphs

Complete graphs are compact.

Every $n \times n$ d.s. matrix is a fractional automorphism of $K_n$. Indeed, let $J$ and $I$ be the all-ones and the identity matrices. Then

$$X \text{ is d.s.} \implies JX = XJ \implies (J-I)X = X(J-I) \implies X \in S(K_n).$$

Birkhoff’s theorem
Every doubly stochastic matrix is a convex combination of permutation matrices.
Basic facts: Closure properties

$G$ is compact iff its complement $\overline{G}$ is compact.

Proof:

$\text{Aut}(G) = \text{Aut}(\overline{G})$ and $S(G) = S(\overline{G})$. Indeed,

$$X \in S(\overline{G}) \iff (J - I - A)X = X(J - I - A) \iff AX = XA \iff X \in S(G'),$$

where $A$ is the adjacency matrix of $G$. 

Lemma (Tinhofer 91)

If a connected graph $G$ is compact, then the $m$-fold disjoint union $mG$ is compact.

Example: The matching graph $mK_2$ and its complement $mK_2$ are compact.
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\[ X \in S(\overline{G}) \iff (J - I - A)X = X(J - I - A) \iff AX = XA \iff X \in S(G), \]

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Further examples of compact graphs

Thus, the cycle graphs $C_3 = K_3$ and $C_4 = 2K_2$ are compact.

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Other examples:

- All cycles [Tinhofer 1986]
- All trees [Tinhofer 1986]
- Many regular graphs [Brualdi 88, Godsil 97, Wang and Li 05]
A negative example

\[ C_3 \cup C_4 \text{ is not compact.} \]
A negative example

$C_3 \cup C_4$ is not compact.

Lemma (Tinhofer 1991)

A regular compact graph is vertex-transitive.

Proof:

- Consider the $n \times n$ all-ones matrix $J$, where $n$ is the number of vertices in $G$.
- If $G$ is regular, then $\frac{1}{n}J \in S(G)$.
- If $G$ is compact, then
  \[
  \frac{1}{n} J = \sum_s \alpha_s P_s,
  \]
  a convex combination of permutation matrices from $\text{Aut}(G)$.
- Therefore, for all $i$ and $j$ there is $s$ such that $[P_s]_{ij} = 1$. 
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Theorem (Arvind, Köbler, Rattan, V. 2015)

All graphs definable in $\text{FO}^2_\#$ are compact.
The proof is based on our characterization of $\text{FO}_2^\#$-definable graphs.
Proof-scheme

• The proof is based on our characterization of $\text{FO}_\#^2$-definable graphs.

Let $G$ be a definable graph. If $X, Y \subseteq V(G)$ are vertex classes in the stable coloring of $G$ (cells), then

• $G[X]$ is one of

$$K_s, \overline{K_s}, mK_2, \overline{mK_2}, \text{ and } C_5.$$ 

• $G[X, Y]$ is one of

$$K_{s,t}, \overline{K_{s+t}}, sK_{1,t}(s \geq 2), \text{ and its bipartite complement.}$$
homogeneous (isotropic) links $G[X, Y]$ can be ignored; 
there is no cycle of non-homogeneous (anisotropic) links; 
each of the corresponding tree-like components of $G$ can be considered separately; 
each such component contains at least one non-homogeneous cell $X$ ($G[X] \cong mK_2, \overline{mK_2}$, or $C_5$); 
induction on the number of cells is possible because fractional automorphisms respect the stable partition [Ramana, Scheinerman, Ullman 1994]; 
the base case is done by the compactness of $K_s, \overline{K_s}, mK_2, \overline{mK_2}$, and $C_5$. 
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Tinhofer’s canonization algorithm for compact graphs

Input: a graph $G$

1. Run Color Refinement on $G$ till color stabilization.
2. If all color classes are singletons, terminate.
3. If there is a color class with 2 or more vertices, individualize one of them by assigning a new color (the lexicographically first unused one).

Theorem (Tinhofer 1991) If an input graph $G$ is compact, then the above algorithm produces a canonical labeling of $G$ for any choice of vertices to be individualized.
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