A logical approach to Isomorphism Testing and Constraint Satisfaction

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Part 6:
Existential-positive FO^2 and Constraint Satisfaction

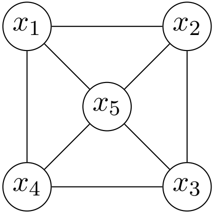
Outline

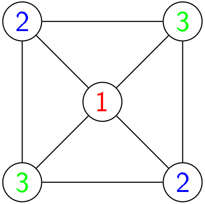
1. Constraint Satisfaction Problem and Constraint Propagation
2. The existential k-pebble game
3. k-Consistency Checking
4. Time complexity of Arc Consistency

Outline

Constraint Satisfaction Problem (CSP)

Variables	x_1, x_2, x_3, x_4, x_5
Values	$x_i \in \{1, 2, 3\}$
Constraints	$x_1 \neq x_2$, $x_2 \neq x_3$, $x_3 \neq x_4$, $x_4 \neq x_1$, $x_1 \neq x_5$, $x_2 \neq x_5$, $x_3 \neq x_5$, $x_4 \neq x_5$
Question:	Is there an assignment of values to the variables satisfying all constraints?





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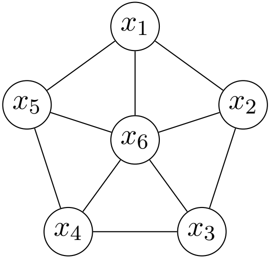
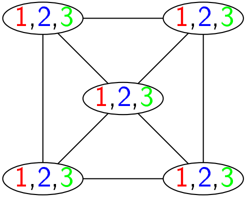


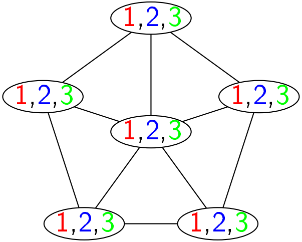
No!

Methodolody: derivation instead of search

Example: 3-COLORABILITY.
We can choose an arbitrary edge and color it arbitrarily.

Derivation rules: $\dfrac{x = 1,\ y \neq x}{y \neq 1}$ etc.





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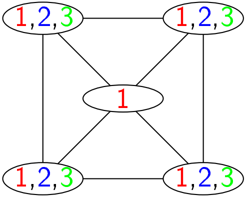

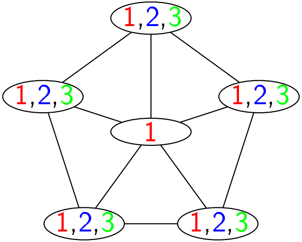

Constraint propagation

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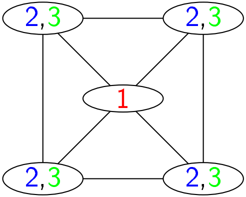

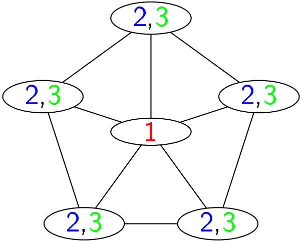

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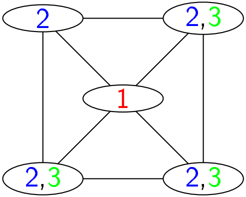

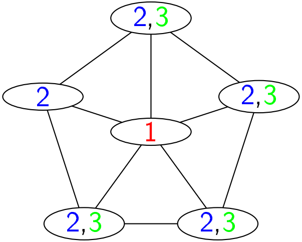

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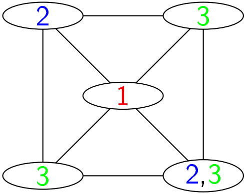

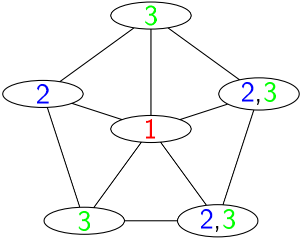

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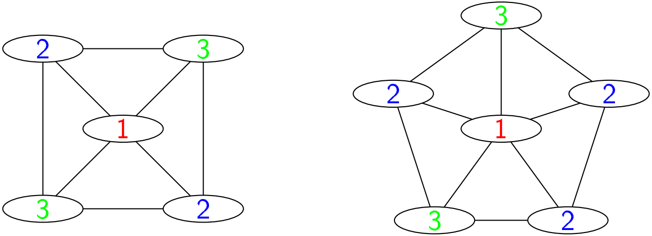
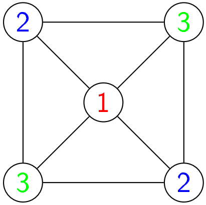
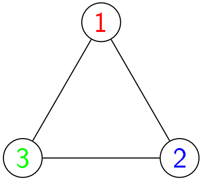


The Feder-Vardi paradigm: a CSP = a Homomorphism Problem

For example, a graph G is 3-colorable iff there is a homomorphism from G to K_3 (notation: $G \to K_3$).





Outline

A logic and a game for the Homomorphism Problem

The following three conditions are equivalent:

- $G \not\to H$,
- some existential-positive formula distinguishes G from H,
- Spoiler has a winning strategy in the existential k-pebble game on G and H for some k.

A logic and a game for the Homomorphism Problem

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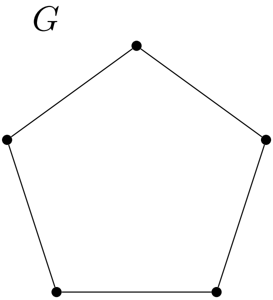
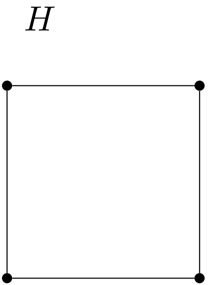
The existential-positive logic $\mathrm{FO}_{\exists,+}$ allows only monotone Boolean connectives (no negation) and only existential quantifiers (no universal quantification).

The existential k-pebble game on G and H is the version of the k-pebble Ehrenfeucht game where

- Spoiler moves always in G,
- Duplicator must keep a partial homomorphism.

The existential k-pebble game (Kolaitis-Vardi)





Spoiler

Objective:
show that $\neg\exists$ homom. $G \to H$

Duplicator

it may exist

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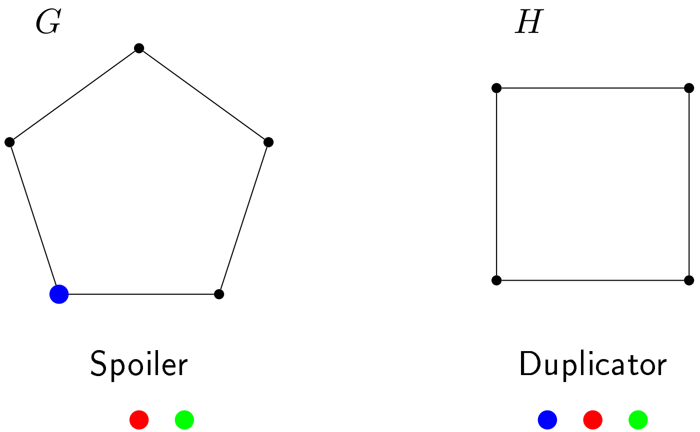

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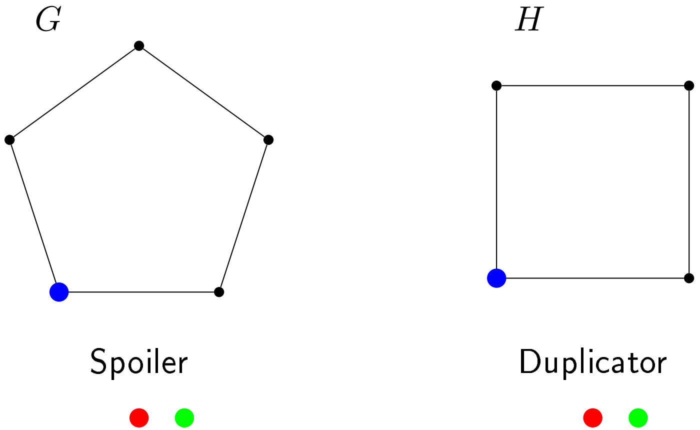

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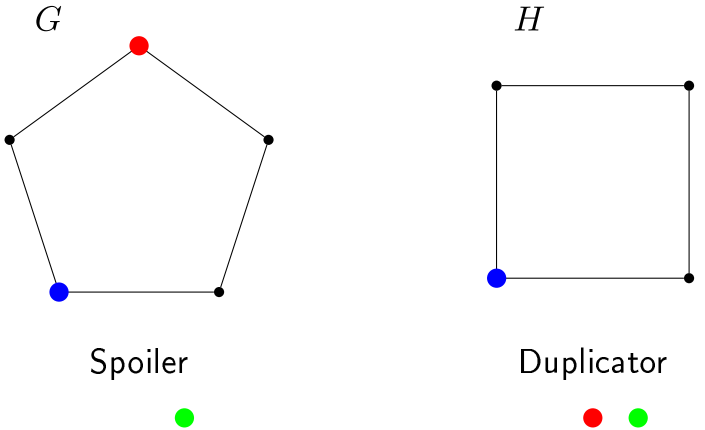

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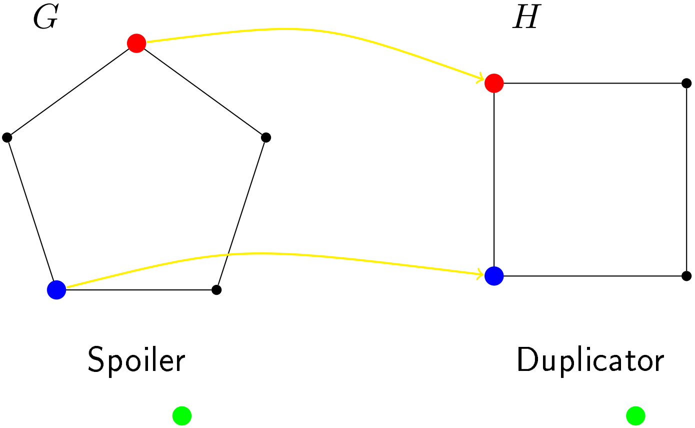

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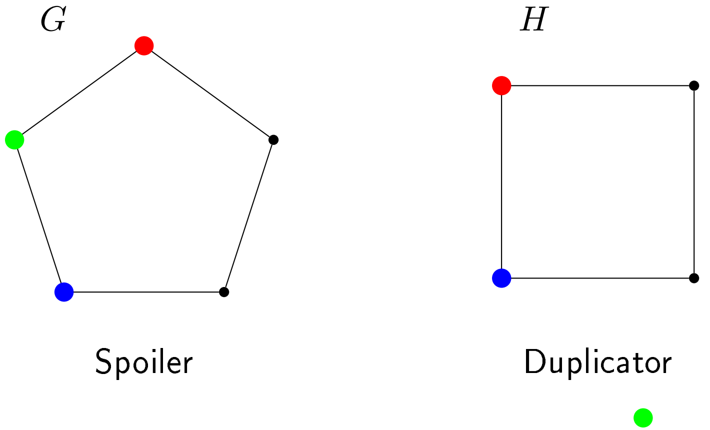

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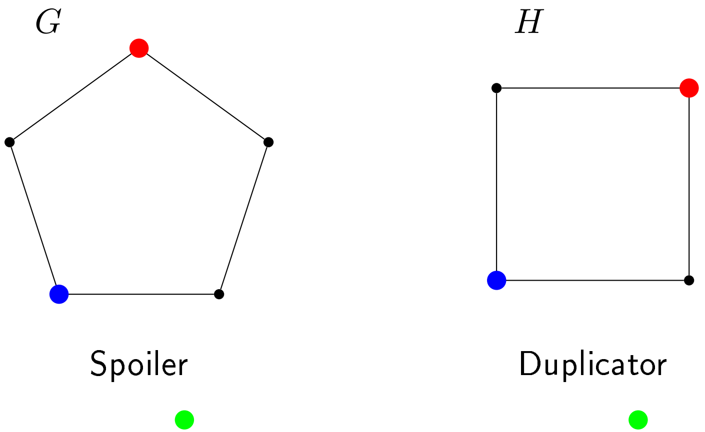

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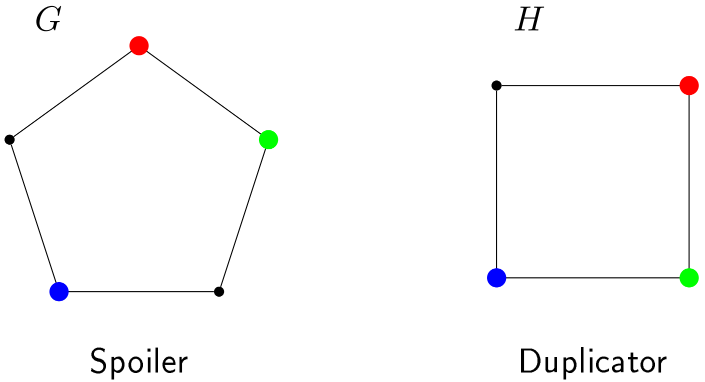

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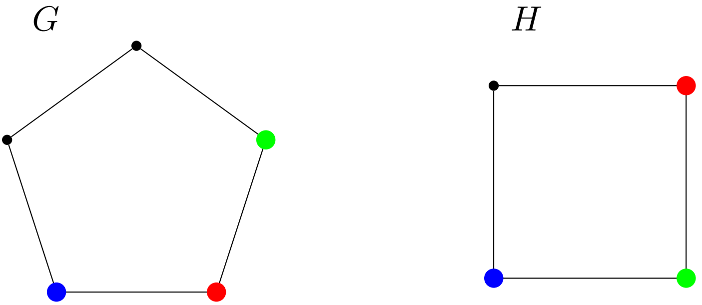


Spoiler

Duplicator

The existential k-pebble game

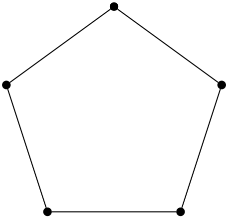
Observation:

If there is a homomorphism h from G to H, then Duplicator wins by pebbling $h(v)$ if Spoiler pebbles v.

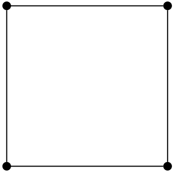
Hence, the game can be used as heuristics for the CSP.

Of course, it is incomplete if k is not large enough.
In our example, $k = 3$ is enough and $k = 2$ is not.

G

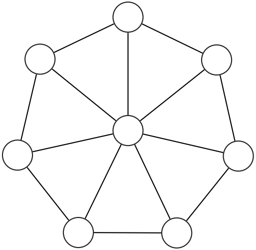


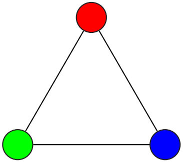
H



Another example

Let G_n denote the wheel graph with n vertices.
If n is even, then Spoiler wins the existential game on G_n and K_3 with 4 pebbles.





Spoiler

1 2 3 4

Duplicator

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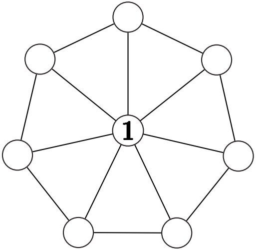

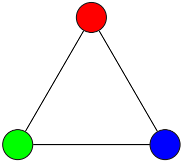

Spoiler
2 3 4

Duplicator
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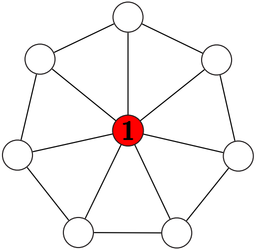

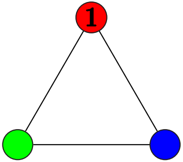

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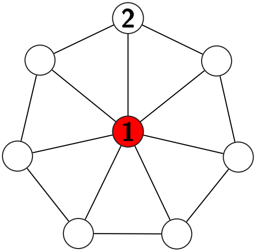

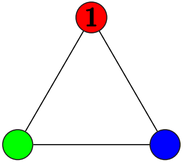

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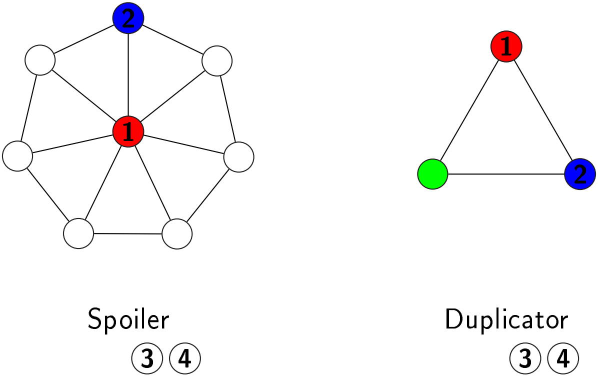

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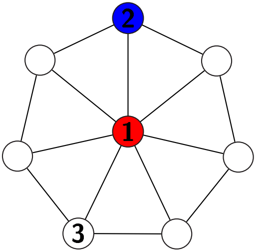

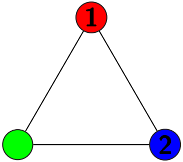

Spoiler

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Duplicator

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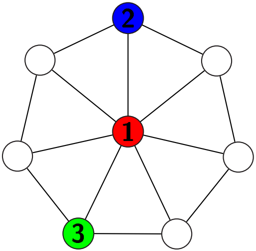

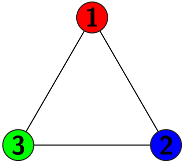

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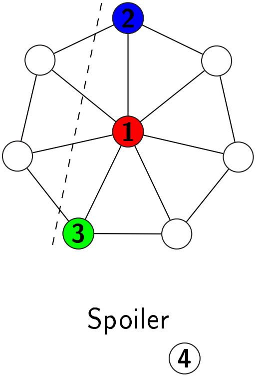

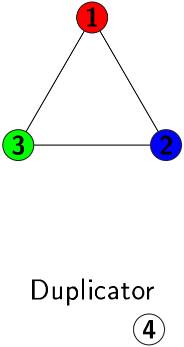

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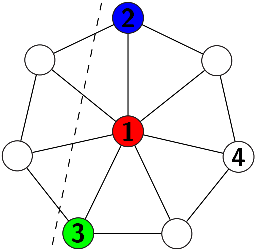

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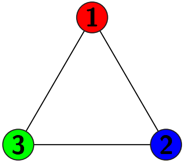

Duplicator

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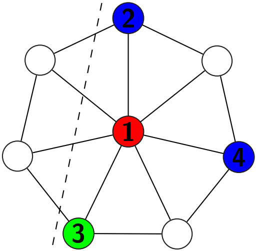

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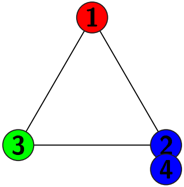

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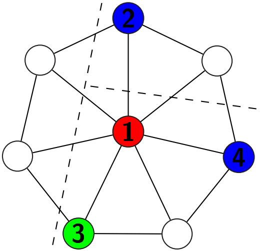

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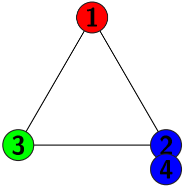

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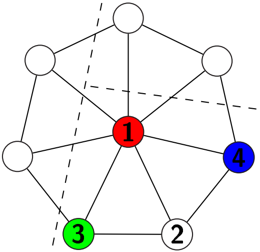

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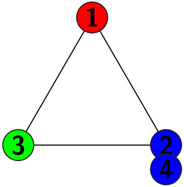


Duplicator

Outline

The triad: A logic, a game, an algorithm

Theorem (Kolaitis, Vardi 95)

Suppose that $G \not\to H$. Then the following three conditions are equivalent:

- $W_{\exists,+}(G,H) \leq k$, i.e., G is distinguishable from H by an existential-positive sentence with k variables;
- Spoiler wins the existential k-pebble game on G and H;
- k-Consistency Checking recognizes that $G \not\to H$.

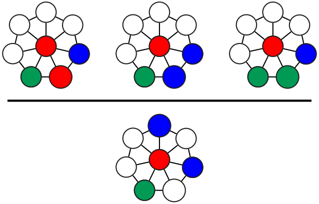
k-Consistency Checking (recasted)

Algorithmic problem

Given two finite structures G and H, does Spoiler win the existential k-pebble game on these structures?

- This is a relaxation of the homomorphism problem.
- For small k, it is commonly used as a heuristics approach.

A propagation-based algorithm makes derivations like



- winning positions for Spoiler

$\Downarrow$

- a winning position too

(a position is a mapping of $\leq k$ vertices from $V(G)$ into $V(H)$)

Spoiler has a winning stategy $\Leftrightarrow$ the uncolored graph is derivable.

The time complexity of k-Consistency Checking

Since there are at most $N = v(G)^k v(H)^k$ positions, all derivations can be generated in time N^{k+1} (the wasteful version of k-consistency checking).

Nevertheless, if k is fixed, this takes polynomial time (while CSP is NP-complete).

The time complexity of k-Consistency Checking

Theorem

The k-Consistency problem is solvable in

- *time $O(v(G)^k v(H)^k) = O(n^{2k})$ for each k* *[Cooper 89]*
- *but not in time $O(n^{\frac{k-3}{12}})$ for $k \geq 15$* *[Berkholz 12]*

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- *but not in time* $O(n^{\frac{k-3}{12}})$ *for* $k \geq 15$ *[Berkholz 12]*

Question. What about Arc Consistency ($k = 2$)?

Remark. If $k = 2$, we consider directed graphs.

The time complexity of k-Consistency Checking

Theorem

The k-Consistency problem is solvable in

- *time* $O(v(G)^k v(H)^k) = O(n^{2k})$ *for each* k [Cooper 89]
- *but not in time* $O(n^{\frac{k-3}{12}})$ *for* $k \geq 15$ [Berkholz 12]

Question. What about Arc Consistency $(k = 2)$?

Remark. If $k = 2$, we consider directed graphs.

In practice: All known Arc Consistency $(k = 2)$ algorithms

- AC-1, AC-3, AC-3.1 / AC-2001, AC-3.2, AC-3.3, AC-3$_d$, AC-4, AC-5, AC-6, AC-7, AC-8, AC-$*$ etc.

are based on constraint propagation.

Outline

Bounds for the propagation approach

Denote the number of vertices and edges by $v(G)$ and $e(G)$ resp.

Theorem (Berkholz, V. 2013)

1. *Arc Consistency is solvable in time* $O(v(G)e(H) + e(G)v(H))$, *which implies* $O(n^3)$ *in terms of* $n = v(G) + v(H)$.
2. *Any propagation-based arc consistency algorithm takes time* $\Omega(n^3)$.

Proof-scheme

- A propagation-based algorithm $\mapsto$ a winning strategy for Spoiler
- The time on input $G, H \mapsto$ the size of the game tree
- In fact, it is enough to show that the optimum depth of the game tree, which is equal to $D_{\exists,+}(G, H)$, is large for some G and H.

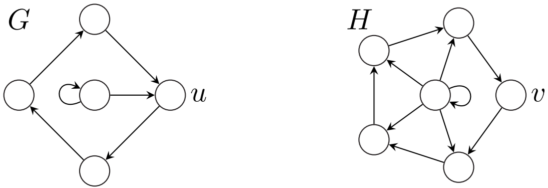
Lemma (constraint propagation can be slow)

There are directed graphs G and H with $v(G) = v(H) - 1 = n$ such that

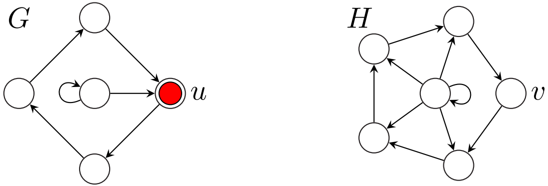
- Spoiler wins the existential 2-pebble game on G and H;
- Duplicator can resist in $\Omega(n^2)$ rounds.

Remark. $n^2 + 1$ rounds always suffice for Spoiler.

An example of slow constraint propagation



An example of slow constraint propagation



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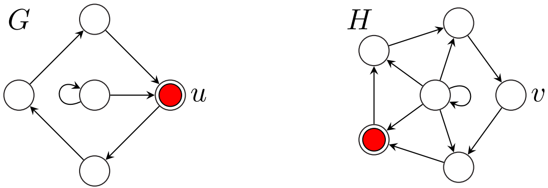

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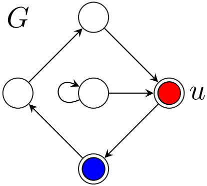

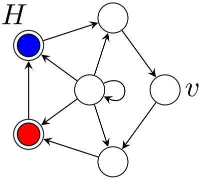

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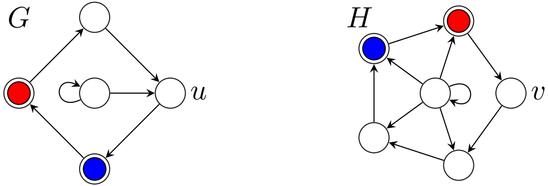

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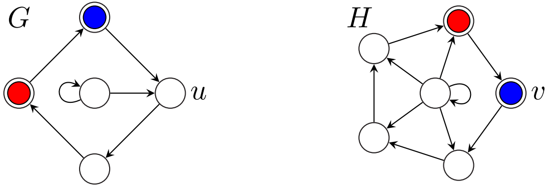

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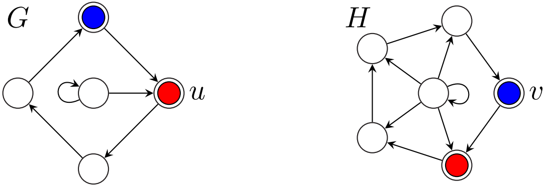

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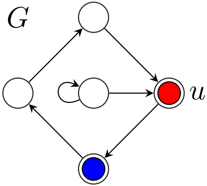

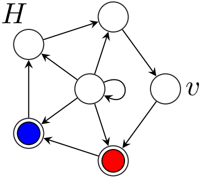

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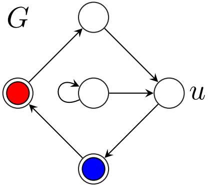

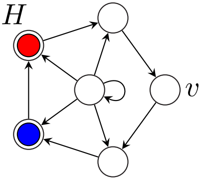

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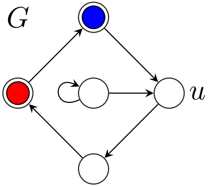

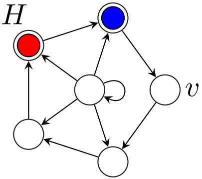

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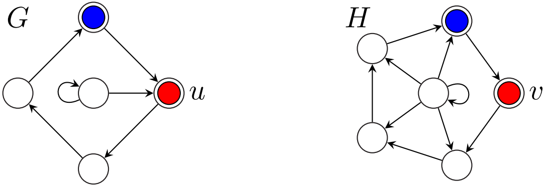

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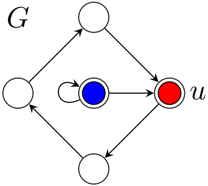

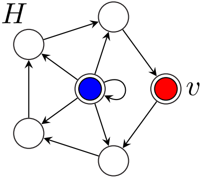

Open problem

Can Arc Consistency be solved faster than in time $O(n^3)$ (by methods different from constraint propagation)?

References

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- C. Berkholz, O. Verbitsky: On the speed of Constraint Propagation and the time complexity of Arc Consistency testing. MFCS'13.