A logical approach to Isomorphism Testing and Constraint Satisfaction

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Part 8: $\mathrm{FO}^k_{\#}$ with $k \geq 3$: Applications to Isomorphism Testing

Outline

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Closure properties of definable graphs

Definition

$\overline{G}$ denotes the *complement graph* of G:
$V(\overline{G}) = V(G)$ and $uv \in E(\overline{G}) \iff uv \notin E(G)$.

Exercise

Prove that, if G is definable in $\mathrm{FO}^k_\#$, then $\overline{G}$ is also definable in $\mathrm{FO}^k_\#$.

Closure properties of definable graphs

Definition

$G_1 + G_2$ denotes the vertex-disjoint union of graphs G_1 and G_2.

Exercise

Let $k \geq 3$. Suppose that connected graphs $G_1, \ldots, G_m$ are definable in $\mathrm{FO}^k_{\#}$. Prove that $G_1 + G_2 + \ldots + G_m$ is also definable in $\mathrm{FO}^k_{\#}$.

Hint

As an example, let $k = 3$ and suppose that $G = 5A + 4B$ and $H = 4A + 5B$ for some connected A and B such that $W_{\#}(A, B) \leq 3$. In the first round, Spoiler makes a counting move in G by marking all vertices in all A-components. Duplicator is forced to mark at least one vertex v in a B-component of H. Spoiler pebbles v, and Duplicator can only pebble some vertex u in an A-component of G. From now on, Spoiler plays in the components that contain u and v using his winning strategy for the game on A and B. If Duplicator marks a vertex in a different component, she loses anyway (why?).

Cographs

Definition

A graph is called a *cograph* if it contains no P_4 as an induced subgraph.

Theorem (Corneil et al.1981)

1. *P_1 is a cograph.*
2. *If G is a cograph, then $\overline{G}$ is also a cograph.*
3. *If G and H are cographs, then $G + H$ is also a cograph.*
4. *A graph is a cograph only if it can be constructed according to the preceding statements.*

Corollary

- *Every cograph is definable in $\mathrm{FO}^3_{\#}$.*
- *The isomorphism problem for cographs is solvable in polynomial time by the 2-dim Weisfeiler-Leman algorithm.*

Outline

k-dimensional Weisfeiler-Leman algorithm

- 1-dim WL $=$ the color refinement algorithm
- k-dim WL colors $V(G)^k$
- Initial coloring: $C^1(\bar{u})$ $=$ the equality type of $\bar{u} \in V(G)^k$ and the isomorphism type of the spanned subgraph
- Color refinement:
  $C^i(\bar{u}) = \{C^{i-1}(\bar{u}), \{(C^{i-1}(\bar{u}^{1,x}), \ldots, C^{i-1}(\bar{u}^{k,x}))\}_{x \in V}\}$,
  where $(u_1, \ldots, u_i, \ldots, u_k)^{i,x} = (u_1, \ldots, x, \ldots, u_k)$

The Weisfeiler-Leman algorithm

- purports to decide if input graphs G and H are isomorphic:
  - If $G \cong H$, the output is correct,
  - if $G \ncong H$, the output can be wrong;
- has two parameters: dimension and number of rounds.
- Fixed dimension $k \implies \leq n^k$ rounds $\implies$ polynomial running time.
- Fixed dimension and $O(\log n)$ rounds $\implies$ parallel logarithmic time.

The Weisfeiler-Leman algorithm

Corollary (Cai, Fürer, Immerman 92)

Let $\mathcal{C}$ be a class of graphs G with $W_{\#}(G) \leq k$ for a constant k. Then Graph Isomorphism for $\mathcal{C}$ is solvable in P.

Corollary (Grohe, V. 06)

1. *Let $\mathcal{C}$ be a class of graphs G with $D^k_{\#}(G) = O(\log n)$. Then Graph Isomorphism for $\mathcal{C}$ is solvable in $\mathrm{TC}^1 \subseteq \mathrm{NC}^2 \subseteq \mathrm{AC}^2$.*
2. *Let $\mathcal{C}$ be a class of graphs G with $D^k(G) = O(\log n)$. Then Graph Isomorphism for $\mathcal{C}$ is solvable in $\mathrm{AC}^1 \subseteq \mathrm{TC}^1$.*

Outline

Classes of graphs: Trees

- $W_{\#}(T) \leq 2$ for every tree T.
- $D^2_{\#}(P_n) \geq \frac{n}{2} - 1$
- Speed-up: one extra variable $\implies$ logarithmic depth !

Theorem

If T is a tree on n vertices, then $D^3_{\#}(T) \leq 3 \log n + 2$.

Proof-sketch

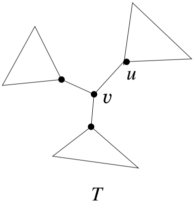
We can easily distinguish between T and $T' \ncong T$ if T'

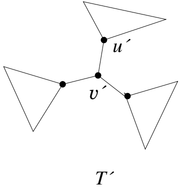
- is disconnected;
- has different number of vertices;
- has the same number of vertices, is connected but has a cycle;
- has larger maximum degree.

It remains the case that T' is a tree with the same maximum degree. For simplicity, assume that the maximum degree is 3 (then no counting quantifiers are needed).

Proof cont'd (a separator strategy)

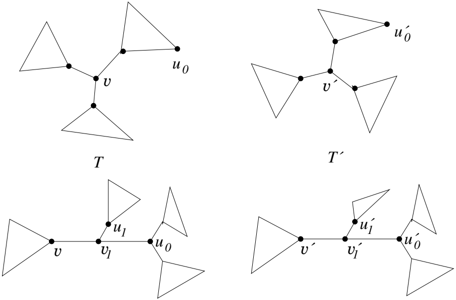
We need to show that Spoiler wins the 3-pebble game on T and T' in $3\log n + 2$ moves.
Step 1. Spoiler pebbles a separator v in T (every component of $T - v$ has $\leq n/2$ vertices).
Step 2. Spoiler ensures pebbling $u \in N(v)$ and $u' \in N(v')$ so that the corresponding components are non-isomorphic rooted trees.





Spoiler forces further play on these components and applies the same strategy again.

A complication: the strategy is now applied to a graph with one vertex pebbled and we may need more than 3 pebbles. Assume that u_0 and u_0' were pebbled earlier and $T - v$ and $T' - v'$ differ only by the components containing u_0 and u_0'. Suppose that $d(v, u_0) = d(v', u_0')$.



Step 3. Spoiler pebbles v_1 in the v-u_0-path such that $T - v_1$ and $T' - v_1'$ differ by components with no pebble (assuming that $d(v, v_1) = d(v', v_1')$).

Isomorphism of trees (history revision)

Theorem

If T is a tree on n vertices, then $D^3_{\#}(T) \leq 3 \log n + 2$.

Testing isomorphism of trees is

- in Log-Space [Lindell 1992]
- in AC^1 [Miller-Reif 1991]
- in AC^1 if $\Delta = O(\log n)$ [Ruzzo 1981]
- in Lin-Time by CR [Edmonds 1965]

Miller and Reif [SIAM J. Comput. 1991]: *"No polylogarithmic parallel algorithm was previously known for isomorphism of unbounded- degree trees."*

However, the $3 \log n$-round 2-WL solves it in TC^1 and is known since 1968 !

Classes of graphs: Bounded tree-width, planar, interval

For a graph G of tree-width k on n vertices
$W_{\#}(G) \le k + 2$ [Grohe, Mariño 99];
$D_{\#}^{4k+4}(G) < 2(k+1)\log n + 8k + 9$ [Grohe, V. 06].

For a planar graph G on n vertices
$W_{\#}(G) = O(1)$ [Grohe 98].
If G is, moreover, 3-connected, then
$D^{15}(G) < 11 \log n + 45$ [V. 07].

For an interval graph G on n vertices
$W_{\#}(G) \le 3$ [Evdokimov et al. 00, Laubner 10];
$D_{\#}^{15}(G) < 9 \log n + 8$ [Köbler, Kuhnert, Laubner, V. 11].

Graphs with an excluded minor

Theorem (Grohe 12)

For each F, if G excludes F as a minor, then

$$W_{\#}(G) = O(1).$$

Open problem

Is it then true that $D^k_{\#}(G) = O(\log n)$ for some constant k?

Outline

References

- J.-y. Cai, M. Fürer, N. Immerman. An optimal lower bound on the number of variables for graph identifications. Combinatorica 12: 389–410 (1992).
- M. Grohe, O. Verbitsky. Testing graph isomorphism in parallel by playing a game. ICALP'06.
- M. Grohe. Fixed-point definability and polynomial time on graphs with excluded minors. J. ACM 59 (2012).

- For practical aspects, see *http://pallini.di.uniroma1.it/*