Model counting for logical theories
Problem set 1

1. Write a propositional formula over 3 variables x_1, x_2, x_3 that evaluates to T if and only if an odd number of variables are set to T.

2. Give an example of sets D and $\mathcal{F}$ such that $\mathcal{F} \subseteq 2^D$ where $(D, \mathcal{F})$ is not a σ-algebra.

3. If $(D, \mathcal{F})$ is a σ-algebra and $A, B \in \mathcal{F}$, show that $A \cap B \in \mathcal{F}$.

4. Give an example where each $(D, \mathcal{F}_i)$ is a σ-algebra, but $(D, \bigcup_i \mathcal{F}_i)$ is not.

   Show that if $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ and each $(D, \mathcal{F}_i)$ is a σ-algebra, then $(D, \bigcup_i \mathcal{F}_i)$ is also a σ-algebra.

   Correction! The second claim above is actually not true. The union $(D, \bigcup_i \mathcal{F}_i)$ is indeed an **algebra** but does **not** have to be a **σ-algebra**. Can you give an example?

5. Let $(D, \mathcal{F}, \mu)$ be a measure space. Show that:

   (a) For $A, B \in \mathcal{F}$, if $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

   (b) For $A, A_1, A_2, \ldots \in \mathcal{F}$, if $A \subseteq \bigcup_{i=1}^{\infty} A_i$, then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

6. Consider the formula $\varphi = x \geq 0 \wedge x \leq 10 \wedge \exists y.\ x = 2y$. What is $\mathsf{mc}(\varphi)$ if

   (a) φ is interpreted in the theory of integer arithmetic?

   (b) φ is interpreted in the theory of real arithmetic?

7. For $n \in \mathbb{N}$ with $n \geq 1$, let c_n be the number of integral points $(x, y) \in \mathbb{Z}^2$ that satisfy the constraint $x^2 + y^2 \leq n^2$, and let s_n be the number of points $(x, y) \in \mathbb{Z}^2$ that satisfy the constraint $x^2 \leq n^2 \wedge y^2 \leq n^2$.

   Does the sequence $\frac{c_1}{s_1}, \frac{c_2}{s_2}, \ldots$ converge to $\frac{\pi}{4}$? Justify your answer.