day 2

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Logical foundations of databases

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Recap

- **Relational model** (tables)
- **Relational Algebra** (union, product, difference, selection, projection)
- **SQL** (SELECT ... FROM ... WHERE ...)
- **RA ≈ basic SQL**
- **First-order logic** (syntax, semantics)
- **Expressiveness:** FO =* RA

Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

$$\begin{aligned} \text{Tables} &= \text{Relations} \\ \text{Rows} &= \text{Tuples} \\ \text{Queries} &= \text{Formulas} \end{aligned}$$

[E.F. Codd 1972]

Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

Tables	=	Relations
Rows	=	Tuples
Queries	=	Formulas

$$\mathrm{RA} =^{*} \mathrm{FO}$$

$$\mathit{How} = \mathit{What}$$

[E.F. Codd 1972]

Formulas as queries

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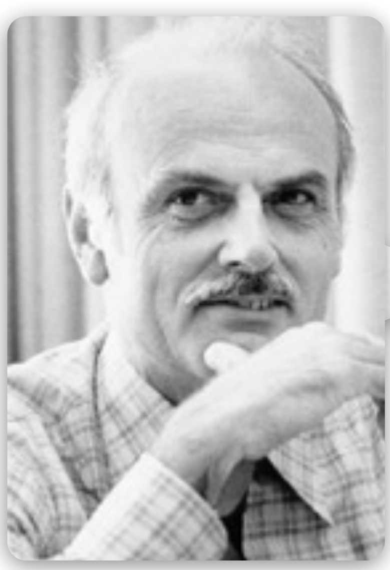
RA $=^*$ FO

How = *What*



RA and FO logic have roughly* the same expressive power!

[E.F. Codd 1972]

*FO without functions, with equality, on finite domains, …

Formulas as queries

$\boxed{\mathrm{RA} \subseteq \mathrm{FO}}$

- $R_1 \times R_2 \quad \rightsquigarrow \quad R_1(x_1, \ldots, x_n) \land R_2(x_{n+1}, \ldots, x_m)$
- $R_1 \cup R_2 \quad \rightsquigarrow \quad R_1(x_1, \ldots, x_n) \lor R_2(x_1, \ldots, x_n)$
- $\sigma_{\{i_1=j_1,\ldots,i_n=j_n\}}(R) \quad \rightsquigarrow \quad R(x_1, \ldots, x_m) \land (x_{i_1}=x_{j_1}) \land \cdots \land (x_{i_n}=x_{j_n})$
- $\pi_{\{i_1,\ldots,i_n\}}(R) \quad \rightsquigarrow \quad \exists(\{x_1,\ldots,x_m\} \setminus \{x_{i_1},\ldots,x_{i_n}\}).\ R(x_1, \ldots, x_m)$
- $R_1 \setminus R_2 \quad \rightsquigarrow \quad R_1(x_1, \ldots, x_n) \land \neg R_2(x_1, \ldots, x_n)$
- …

Formulas as queries

$\boxed{\mathrm{FO} \subseteq \mathrm{RA}}$ does not hold in general!

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elements in the relations

⇝ We restrict variables to range over **active domain**

FO$^{\text{act}}$

=

FO restricted
to active domain

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FO^{act}

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FO restricted
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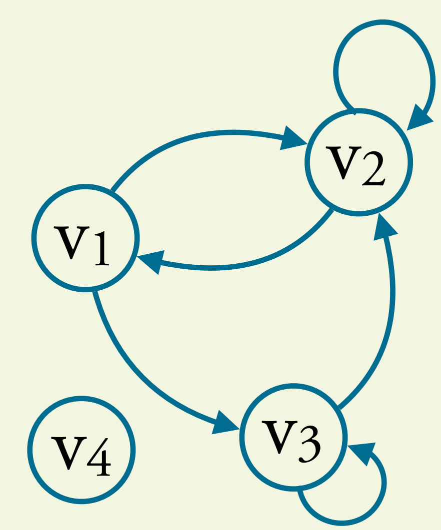
$\phi_1(x) = \forall y\, E(y,x)$
$\phi_1(G) = \{v_2\}$

G = 

$\phi_2(x,y) = \neg E(x,y)$
$\phi_2(G) = \{(v_1,v_1),(v_3,v_1),(v_2,v_3)\}$

First-order logic restricted to active domain

Formal Semantics of FO^{act}

$G \models_\alpha \exists x\, \phi$ iff for **some** $v \in ACT(G)$ and $\alpha' = \alpha \cup \{x \mapsto v\}$ we have $G \models_{\alpha'} \phi$

$G \models_\alpha \forall x\, \phi$ iff for **every** $v \in ACT(G)$ and $\alpha' = \alpha \cup \{x \mapsto v\}$ we have $G \models_{\alpha'} \phi$

$G \models_\alpha \phi \wedge \psi$ iff $G \models_\alpha \phi$ and $G \models_\alpha \psi$

$G \models_\alpha \neg\phi$ iff it is not true that $G \models_\alpha \phi$

$G \models_\alpha x=y$ iff $\alpha(x)=\alpha(y)$

$G \models_\alpha E(x,y)$ iff $(\alpha(x),\alpha(y)) \in E$

$ACT(G) = \{v \mid$ **for some** v': $(v,v') \in E$ or $(v',v) \in E\}$

First-order logic restricted to active domain

$\boxed{\mathrm{FO}^{\mathrm{act}} \subseteq \mathrm{RA}}$

First-order logic restricted to active domain

$FO^{act} \subseteq RA$

Assume:

1. ϕ in normal form: $(\exists^* (\neg\exists)^*)^*$ + quantifier-free $\psi(x_1,\ldots,x_n)$
2. ϕ has n variables

$$\exists x_1\, \exists x_2\, \neg\exists x_3\, \exists x_4\, .\, (\, E(x_1,x_3) \wedge \neg E(x_4,x_2)\,) \vee (x_1 = x_3)$$

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Adom = RA expression for active domain = "$\pi_1(E) \cup \pi_2(E)$"

Translation

- $(R(x_{i_1},\ldots,x_{i_t}))^+ \rightsquigarrow R$
- $(x_i = x_j)^+ \rightsquigarrow \sigma_{\{i=j\}}(\,\mathbf{Adom} \times \cdots \times \mathbf{Adom}\,)$
- $(\psi_1 \wedge \psi_2)^+ \rightsquigarrow \psi_1^+ \cap \psi_2^+$
- $(\neg\psi)^+ \rightsquigarrow \mathbf{Adom} \times \cdots \times \mathbf{Adom} \setminus \psi^+$
- $(\,\exists x_i\, \phi(x_{i_1},\ldots,x_{i_t})\,)^+ \rightsquigarrow \pi_{\{i_1,\ldots,i_t\}\setminus\{i\}}(\,\phi^+\,)$

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$\pi_{\{1,\ldots,n\}}(\sigma_{\{i_1=n+1,\ldots,i_t=n+t\}}(\mathrm{Adom}^n \times R))$

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$A \cap B = ((A \cup B) \setminus (A \setminus B))$
$\setminus (B \setminus A)$

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Adom^t if t is the arity of ψ^+

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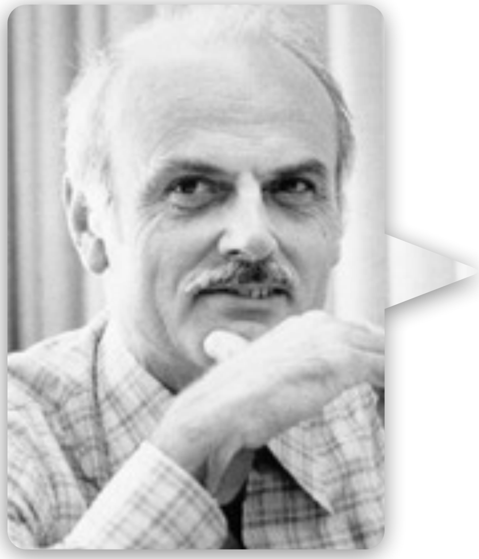
Corollary



$\mathbf{FO}^{\mathrm{act}}$ is equivalent to RA

Question 1: How is $\pi_2(\sigma_{1=3}(R_1 \times R_2)$ expressed in FO?
Remember: R_1,R_2 are binary

Question 2: How is $\exists y,z \,.\, (R_1(x,y) \wedge R_1(y,z) \wedge\ x \neq z\,)$ expressed in RA?
Remember: The signature is the same as before (R_1,R_2 binary)

- $R_1 \cup R_2$
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- $R_1 \setminus R_2$
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) \coloneqq \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$
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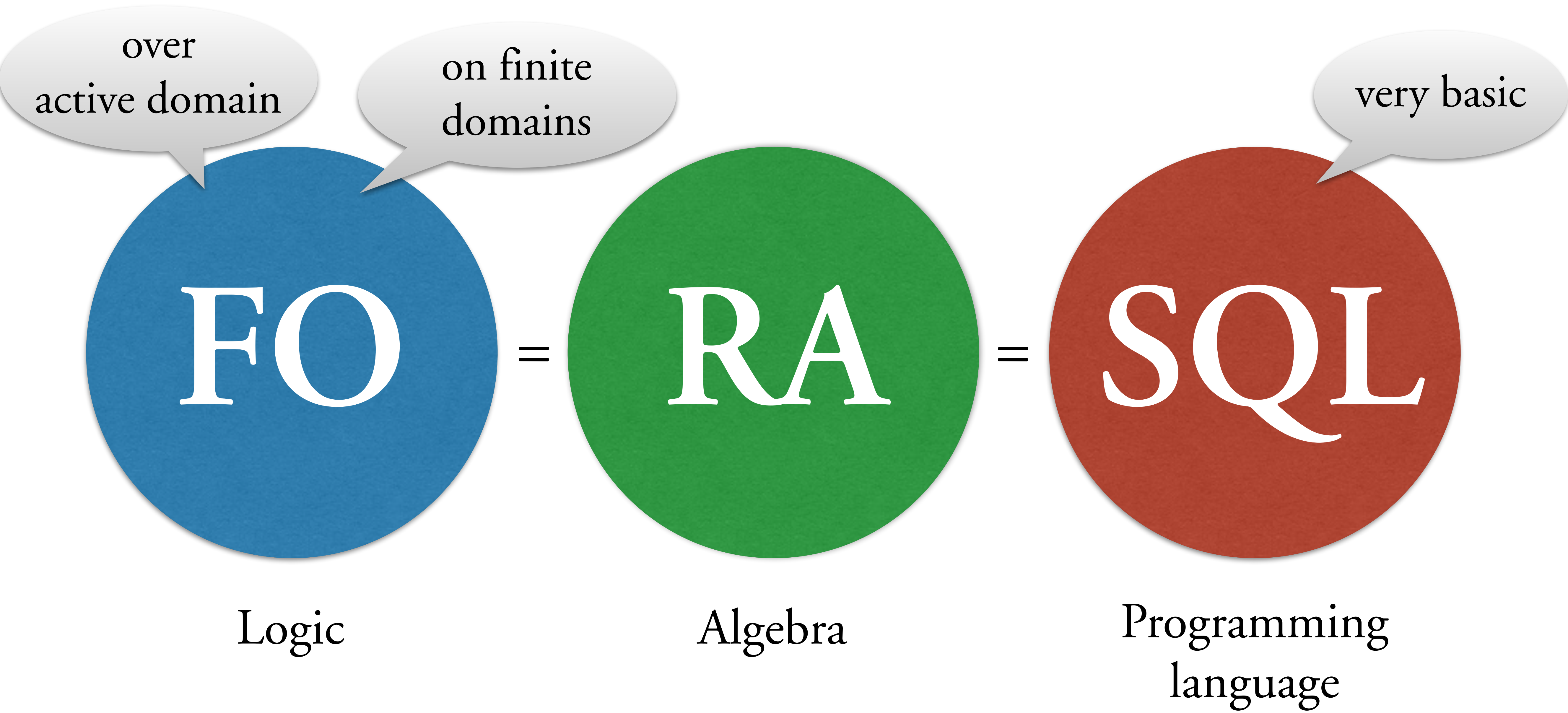
Answer: $\pi_1(\sigma_{\{2=3,1\neq 4\}}(R_1 \times R_1))$

FO = RA = SQL

Logic

Algebra

Programming language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Algorithmic problems for query languages

Evaluation problem: Given a query **Q**, a database instance **db**, and a tuple **t**, is **t** $\in$ **Q**(**db**) ?

⇝ How hard is it to retrieve data?

Algorithmic problems for query languages

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Emptiness problem: Given a query **Q**, is there a database instance **db** so that $\mathbf{Q}(\mathbf{db}) \neq \varnothing$?

⇝ Does Q make sense? Is it a contradiction? (Query optimization)

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Equivalence problem: Given queries $\mathbf{Q}_1$, $\mathbf{Q}_2$, is

$$\mathbf{Q}_1(\mathbf{db}) = \mathbf{Q}_2(\mathbf{db})$$

for all database instances **db**?

⇝ Can we safely replace a query with another? (Query optimization)

Complexity theory

What can be **mechanized**? $\rightsquigarrow$ decidable/undecidable

How **hard** is it to mechanise? $\rightsquigarrow$ complexity classes

Complexity theory

H's 10th PCP
Domino K
$\vdots$

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Domino

H's 10th

PCP

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usage of resources:
- time
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Complexity theory

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Alg(*input*) uses less than $f(|input|)$ units of TIME.

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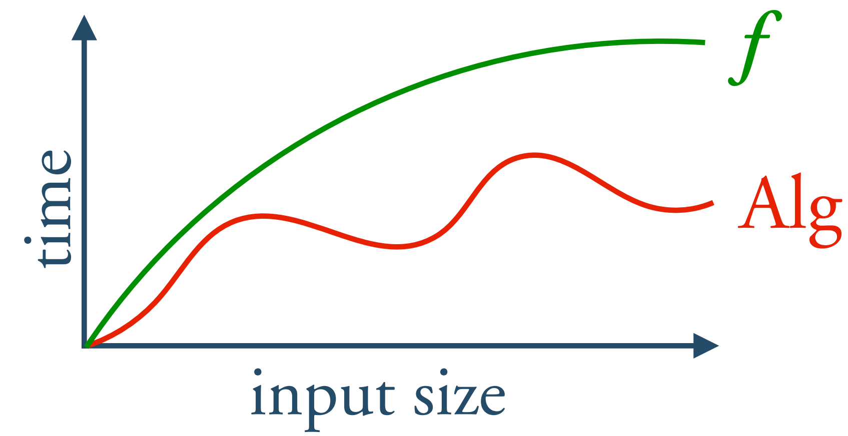

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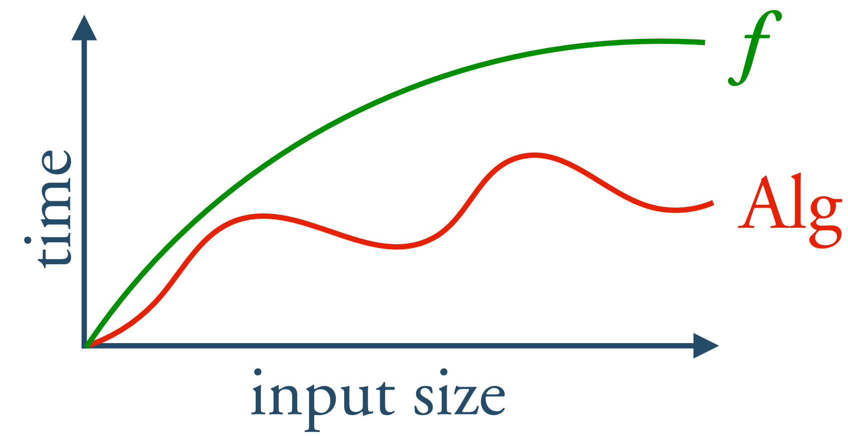

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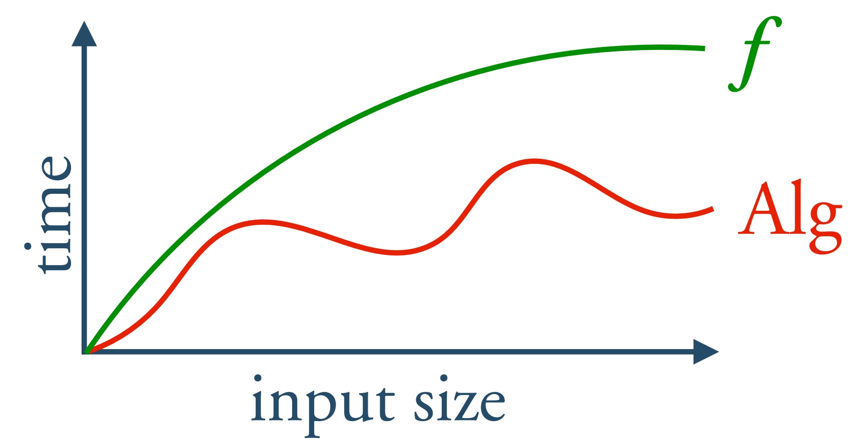

LOGSPACE $\subseteq$ PTIME $\subseteq$ PSPACE $\subseteq$ EXPTIME $\subseteq \cdots$

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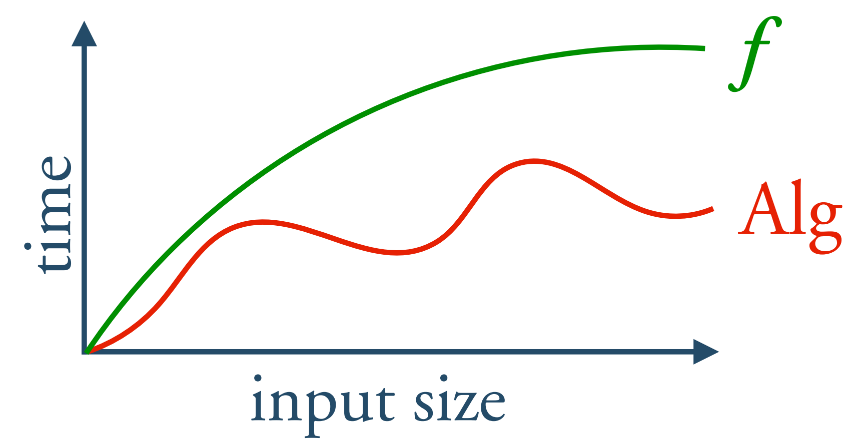

LOGSPACE ⊆ PTIME ⊆ PSPACE ⊆ EXPTIME ⊆ · · ·

PTIME: TIME-bounded by a polynomial

PSPACE: SPACE-bounded by a polynomial

LOGSPACE: SPACE-bounded by log(n)

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, \ldots, x_n)$, a graph G, and a binding α, does $G \models_\alpha \phi$?

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

Equivalence problem: Given FO formulae ϕ,ψ, is

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for all graphs G and bindings α?

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DECIDABLE ⇝ foundations of the database industry

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💀 UNDECIDABLE ⇝ both for $\models$ and $\models_{\text{finite}}$

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💀 UNDECIDABLE $\rightsquigarrow$ by reduction to the satisfiability problem

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Proof: By reduction from the Domino (aka Tiling) problem.

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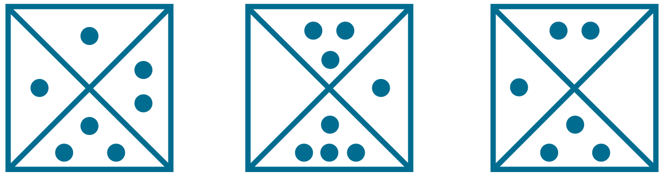
Proof: By reduction from the Domino (aka Tiling) problem.

Reduction from P to P': Algorithm that solves P using a O(1) procedure
" P'(x) "
that returns the truth value of P'(x).

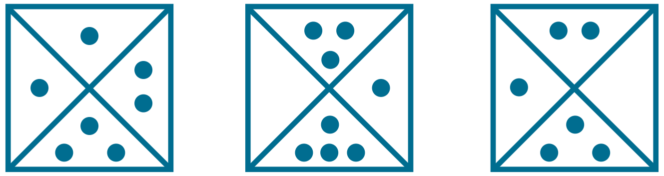
The (undecidable) Domino problem

Domino

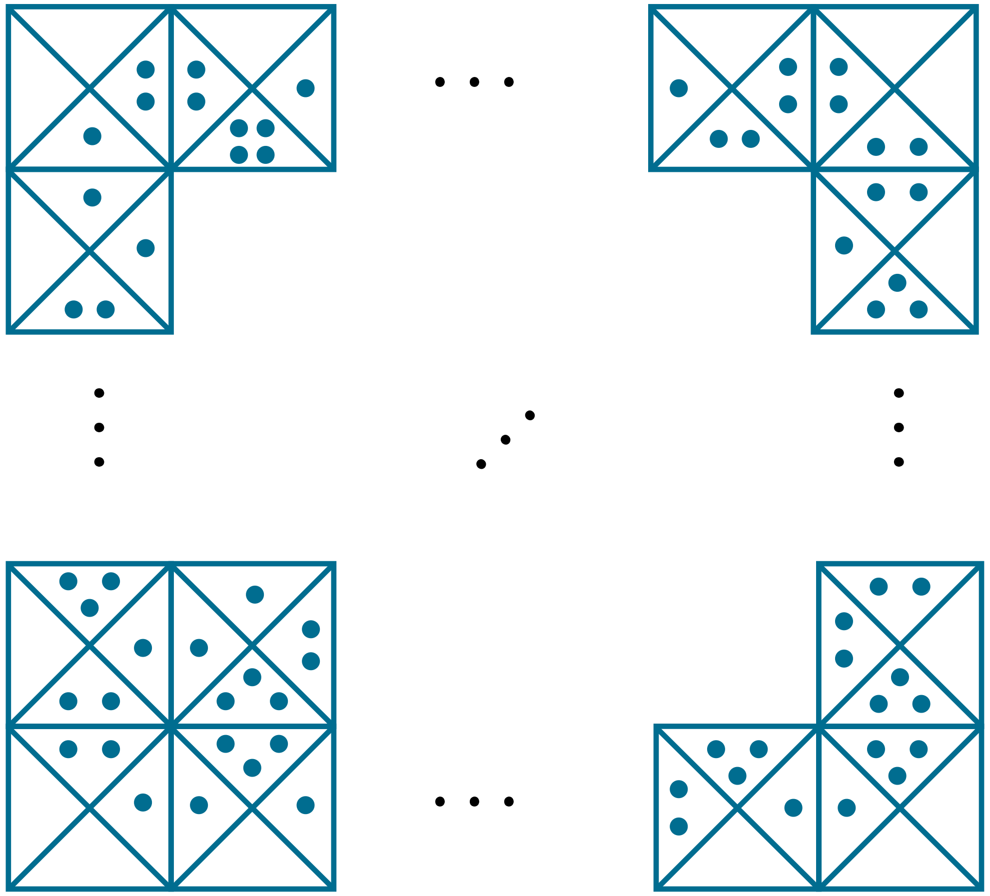
Input: 4-sided dominos: 

The (undecidable) Domino problem

Domino

Input: 4-sided dominos: 

Output: Is it possible to form a white-bordered rectangle? (of any size)



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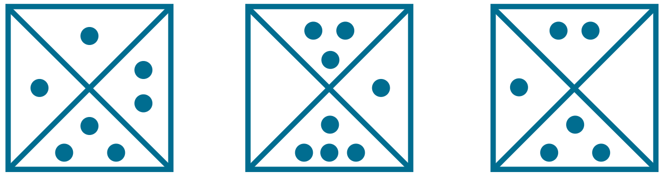
Input: 4-sided dominos: 

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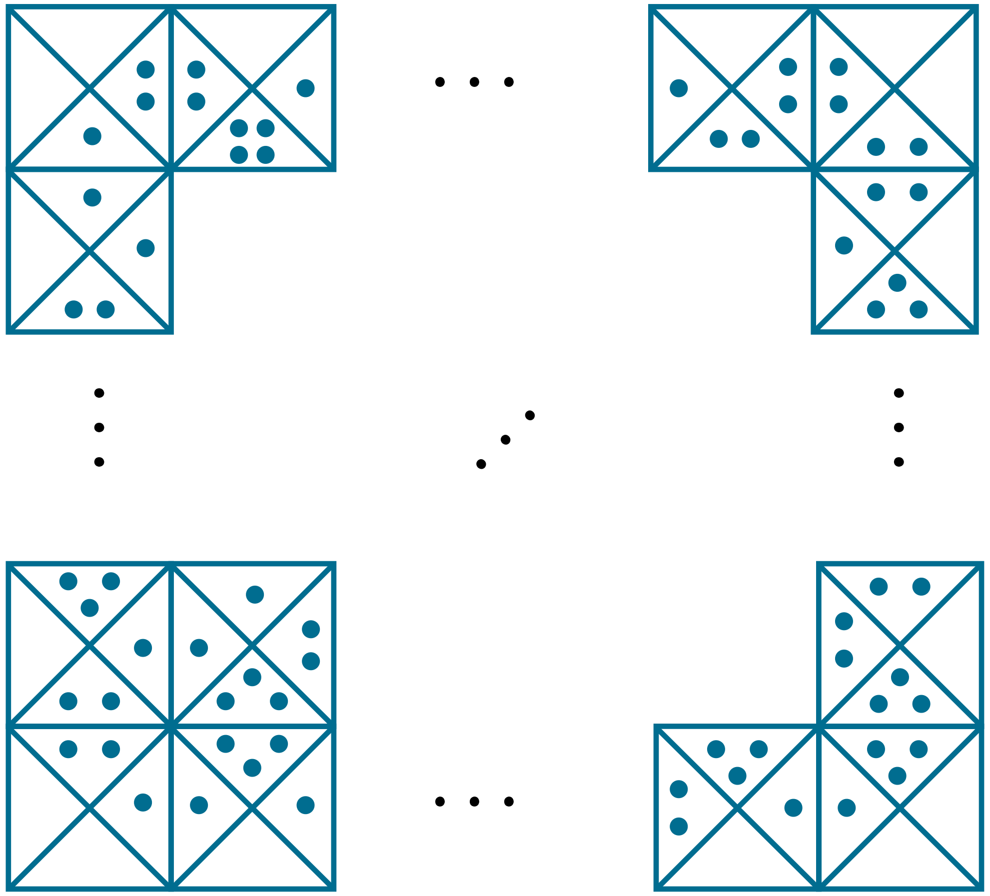
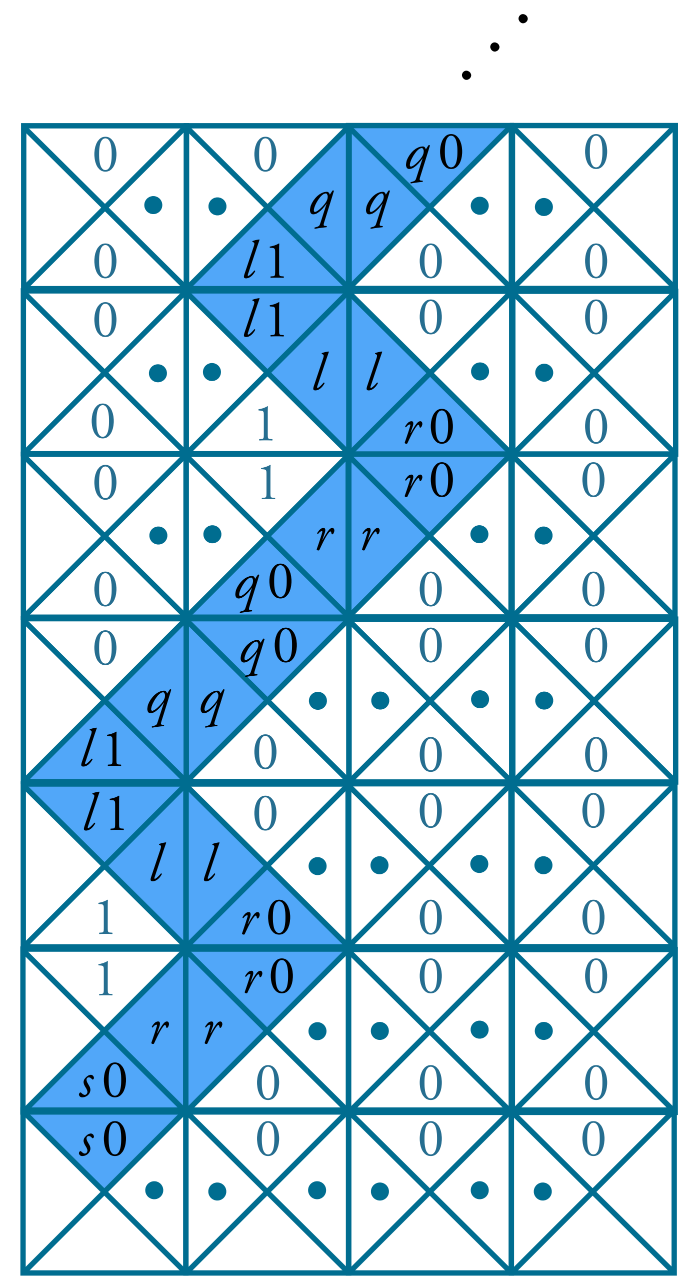


Rules: sides must match,
you can't rotate the dominos, but you can 'clone' them.

Domino - Why is it undecidable?

It can easily encode *halting* computations of Turing machines:



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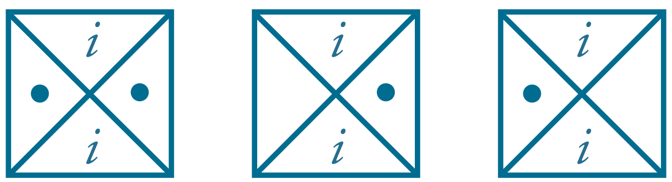

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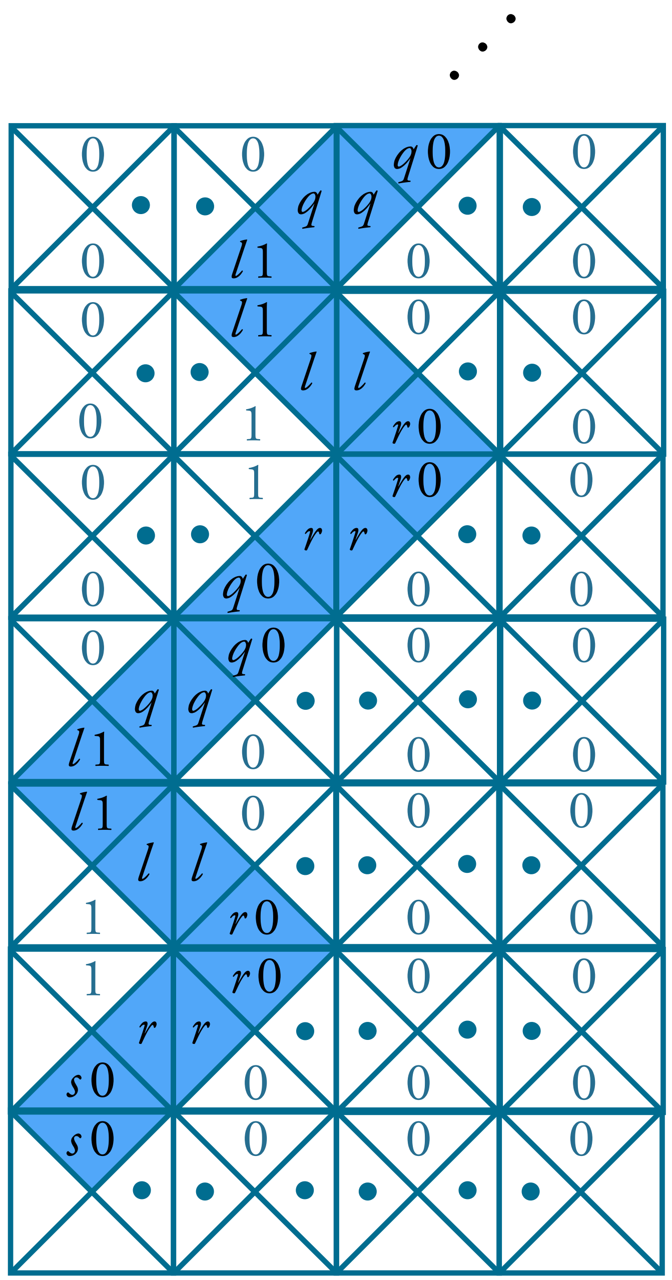

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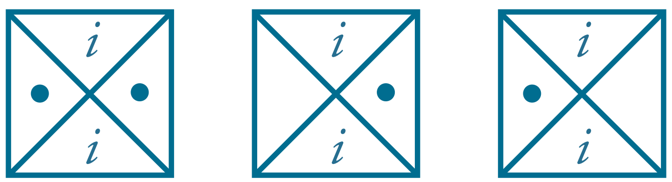

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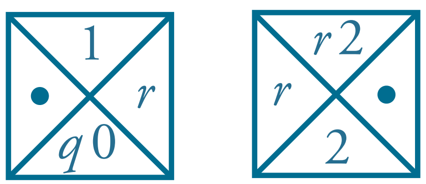

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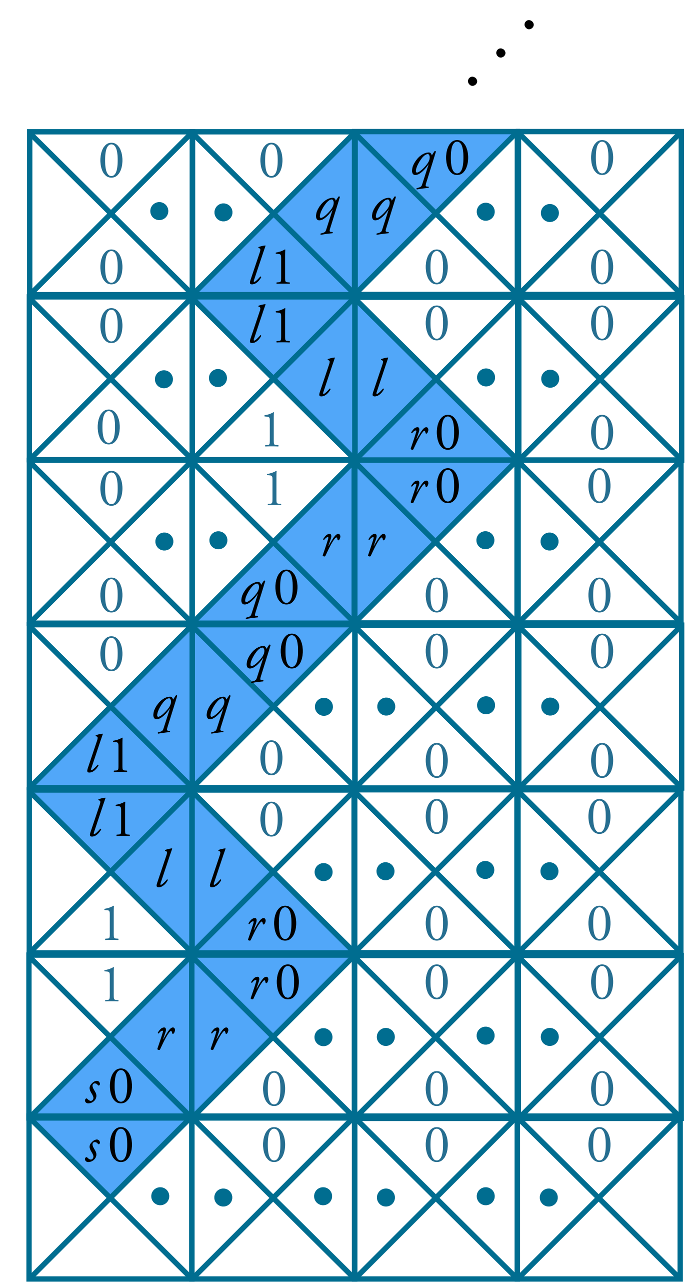

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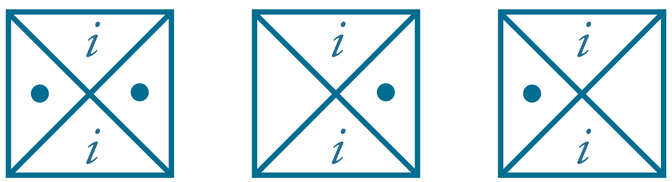

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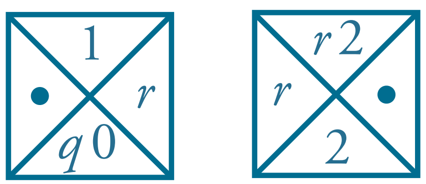

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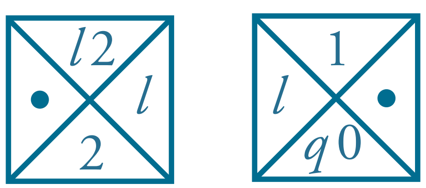

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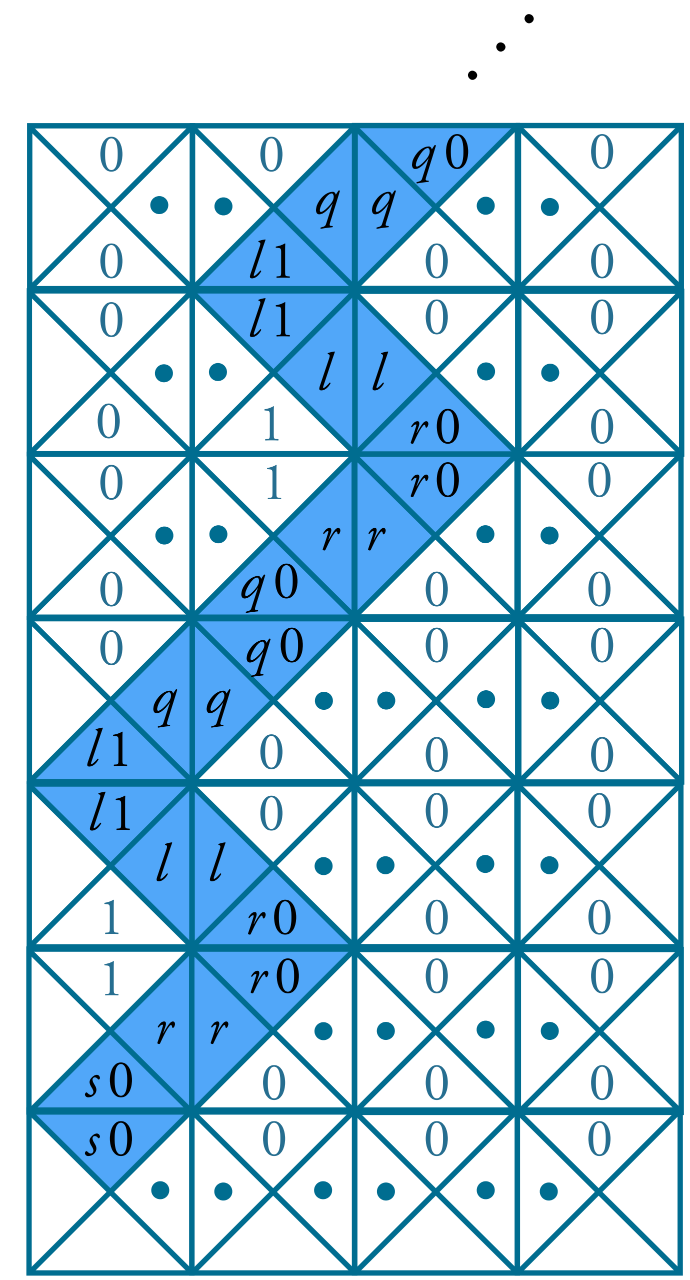

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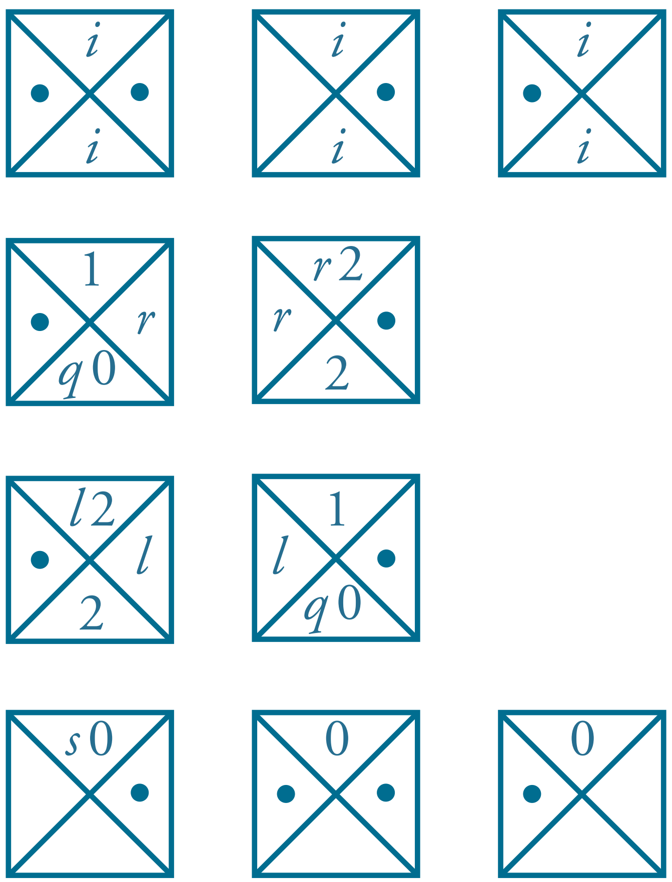

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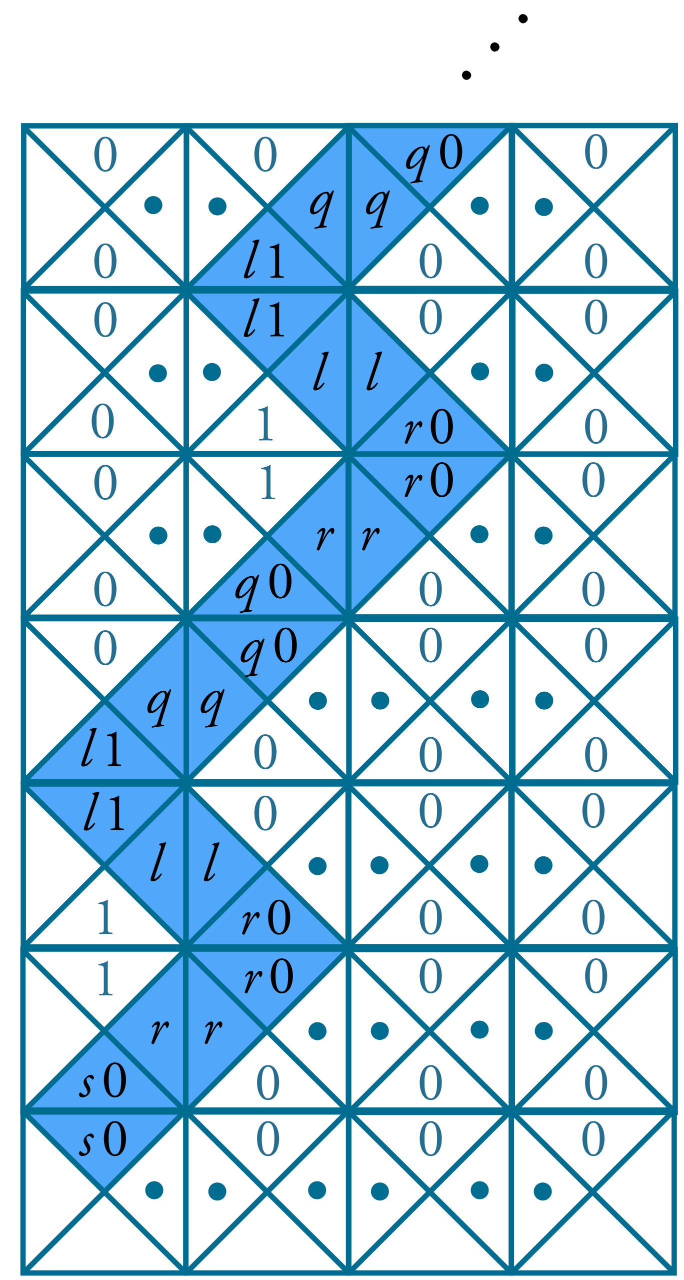

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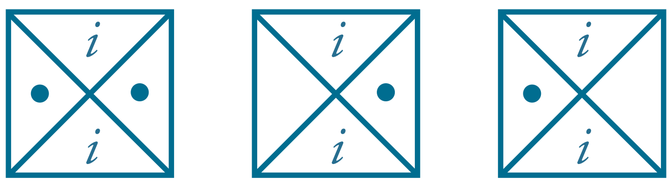

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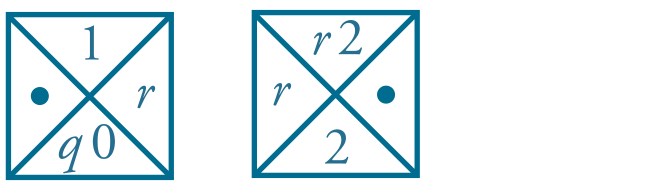

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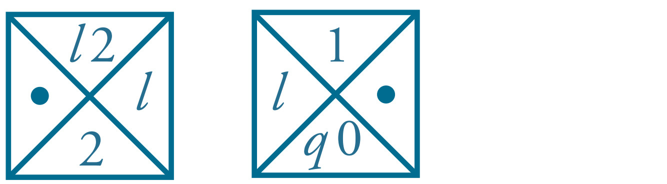

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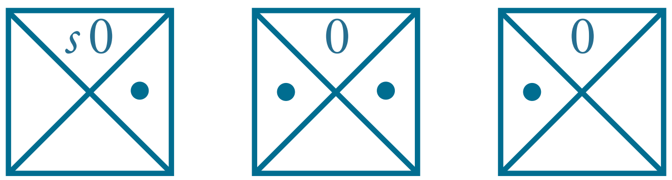

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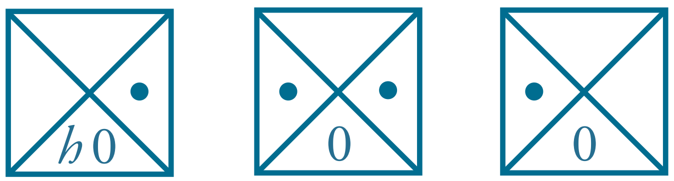

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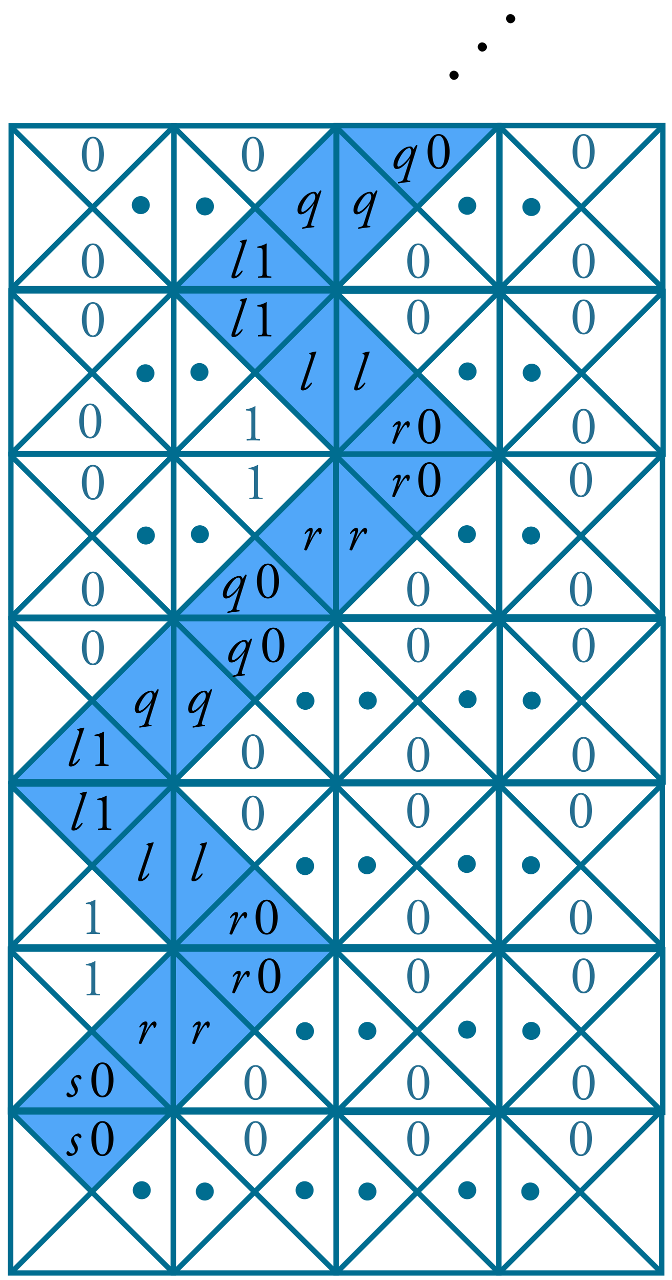
. . .



Domino $\rightsquigarrow$ Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** $H(,)$ and $V(,)$ are relations representing bijections such that…

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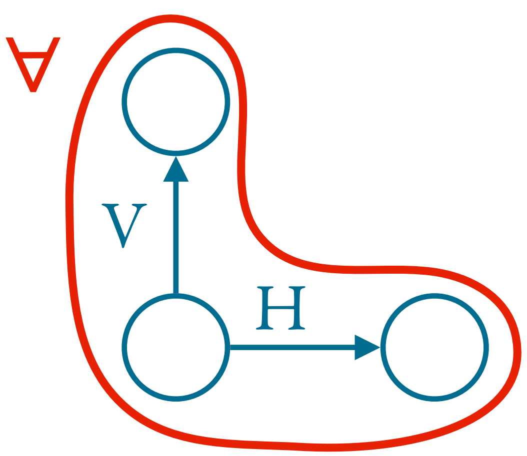

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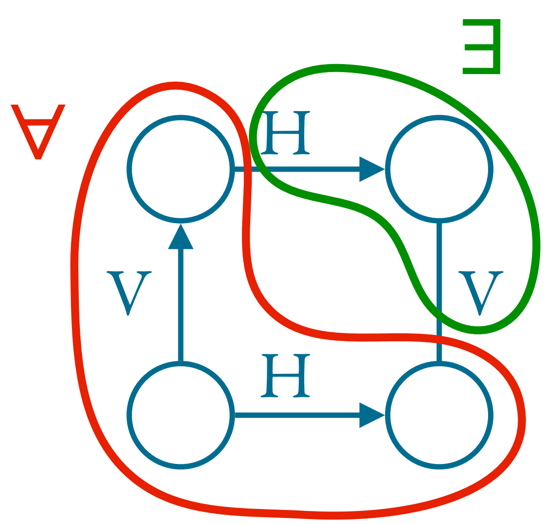

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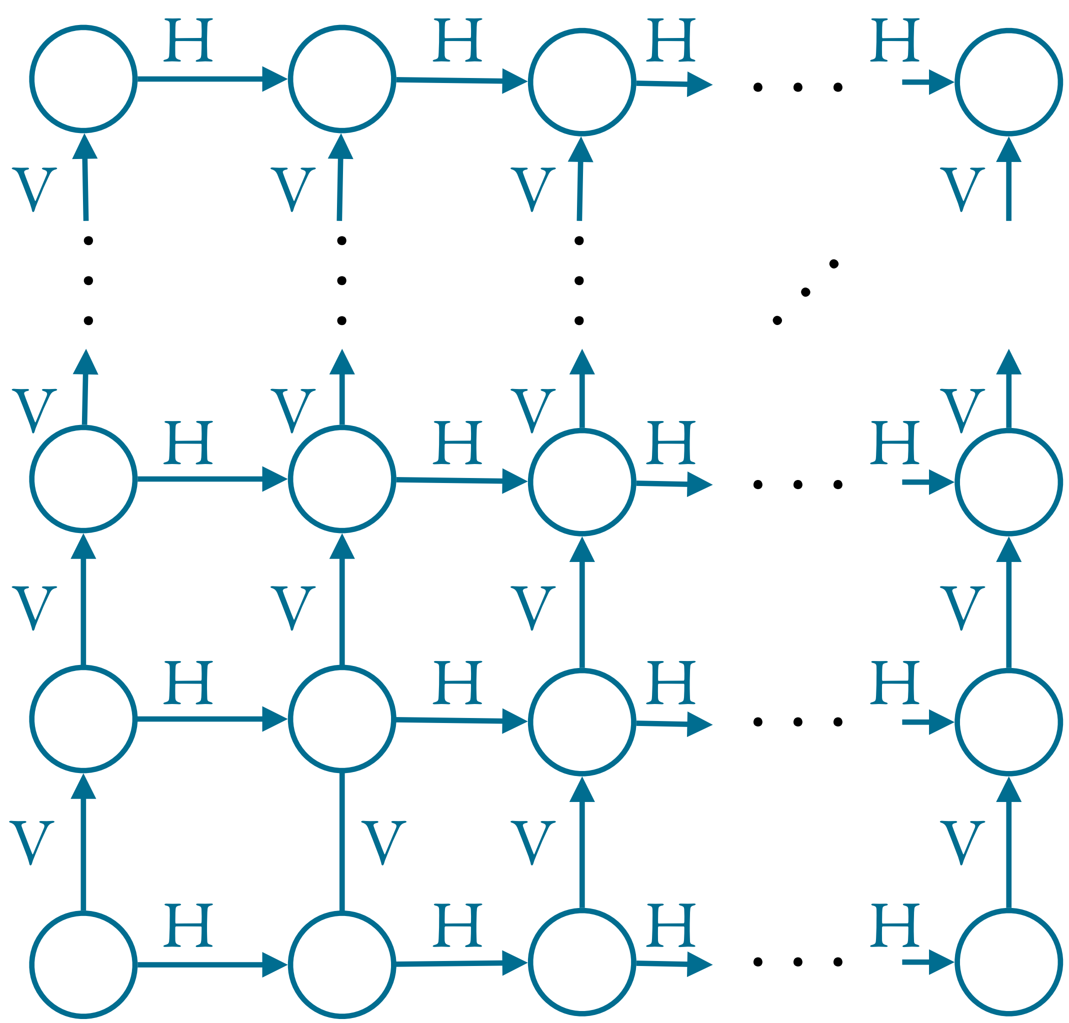

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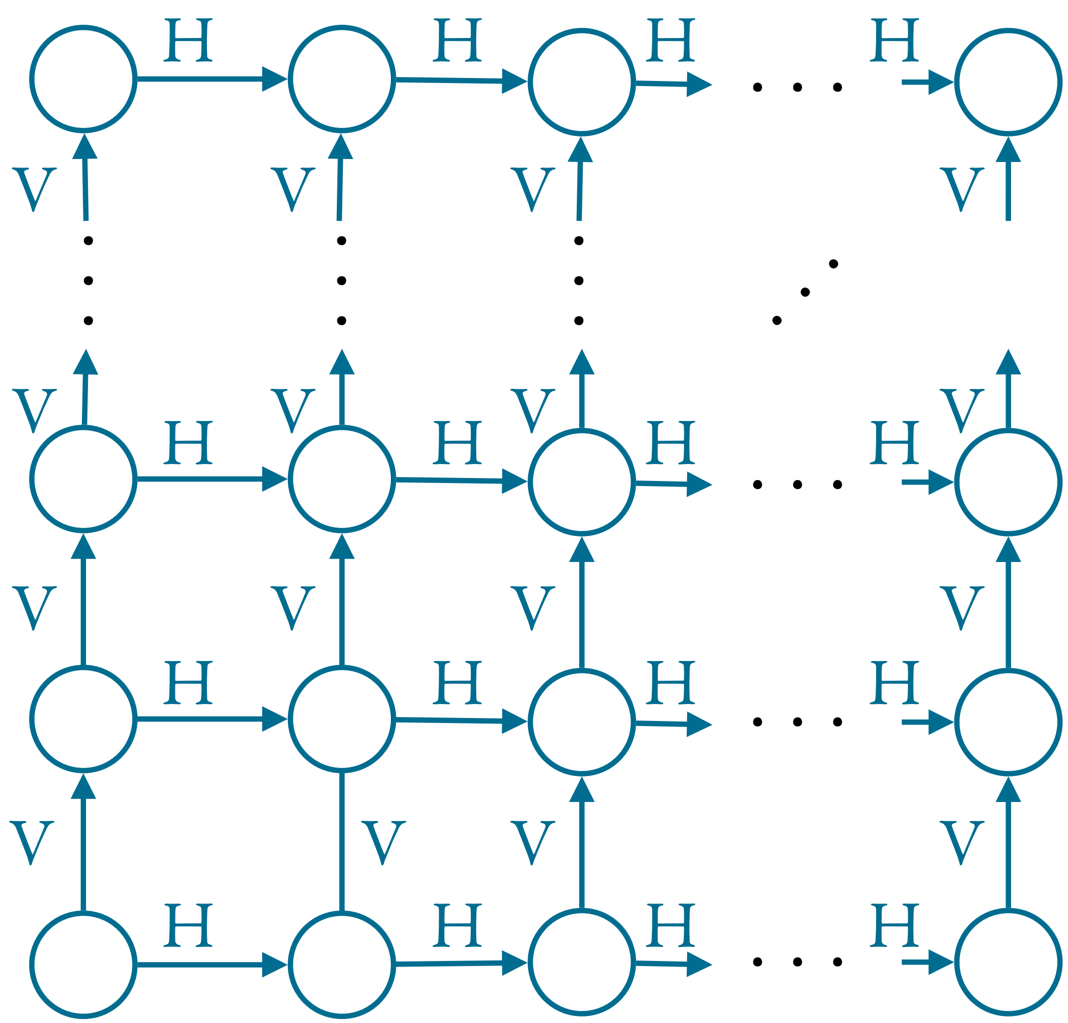

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a unary relation

$$\mathbf{D}_{\boxtimes}(\mathbf{x})$$

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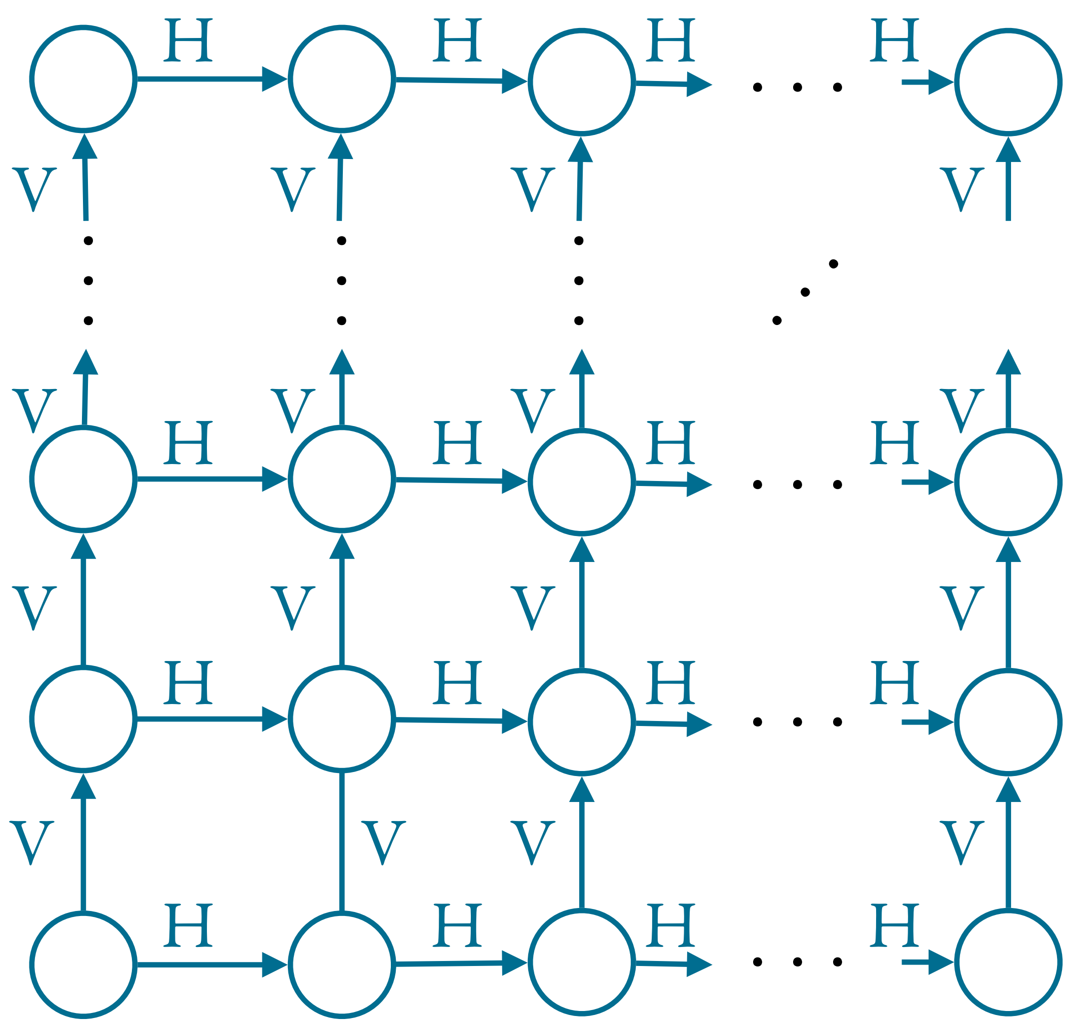

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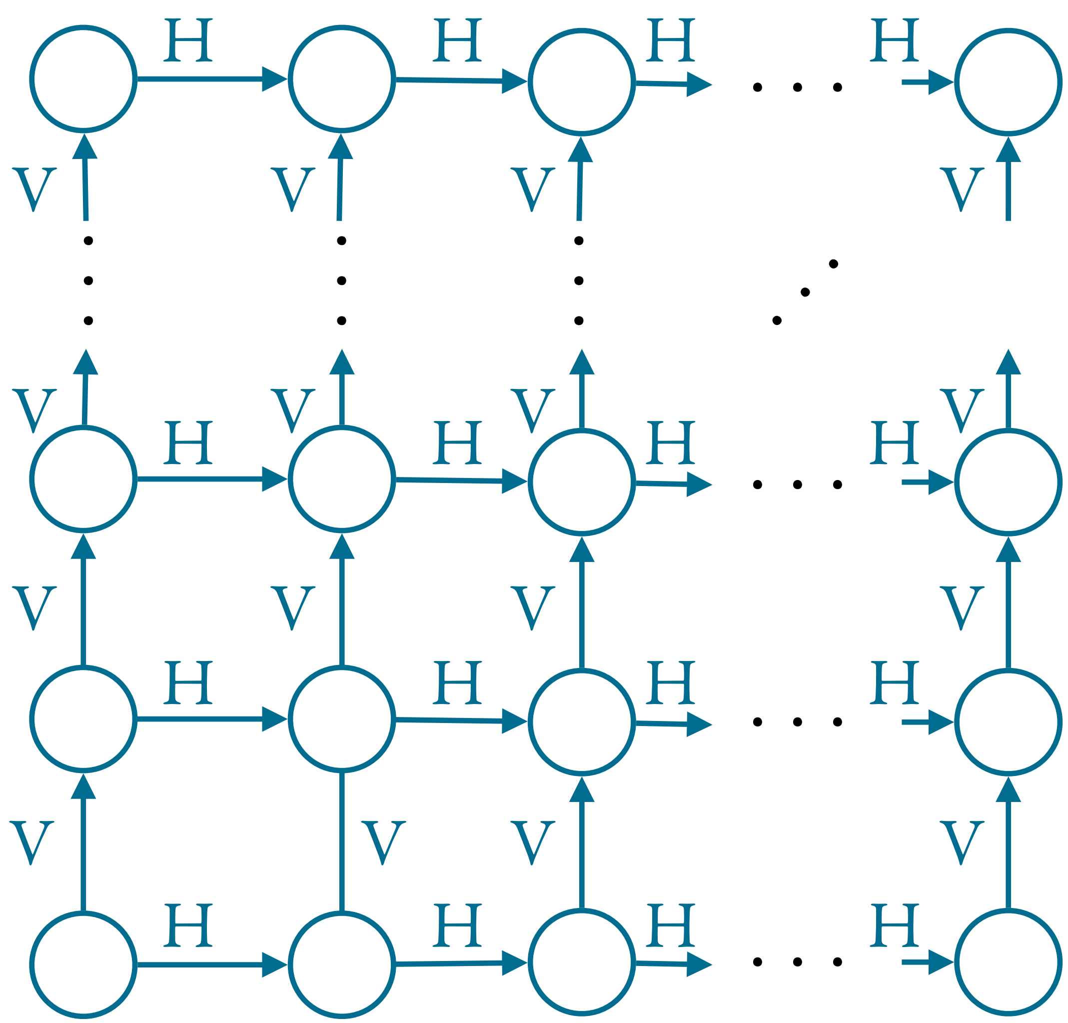

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4. **Borders are white.**

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, \ldots, x_n)$, a graph G, and a binding α, does $G \models_\alpha \phi$?

DECIDABLE ⇝ foundations of the database industry

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

💀 UNDECIDABLE ⇝ both for $\models$ and $\models_{\text{finite}}$

Equivalence problem: Given FO formulae ϕ,ψ, is

$$G \models_\alpha \phi \quad \text{iff} \quad G \models_\alpha \psi$$

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Actually, there are reductions in both senses:

$\phi(x_1,\ldots,x_n)$ and $\psi(y_1,\ldots,y_m)$ are **equivalent** iff

- $n=m$
- $(x_1=y_1) \wedge \cdots \wedge (x_n=y_n) \wedge \phi(x_1,\ldots,x_n) \wedge \neg\psi(y_1,\ldots,y_n)$ is unsatisfiable
- $(x_1=y_1) \wedge \cdots \wedge (x_n=y_n) \wedge \psi(x_1,\ldots,x_n) \wedge \neg\phi(y_1,\ldots,y_n)$ is unsatisfiable

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Input:

$$\left(\begin{array}{l} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{array} \right.$$

Output: $G \models_\alpha \phi$?

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Encoding of $G = (V, E)$

- each node is coded with a bit string of size $\log(|V|)$,
- edge set is encoded by its tuples, e.g. (100,101), (010, 010), …

Cost of coding: $||G|| = |E| \cdot 2 \cdot \log(|V|) \approx |V|$ (mod a polynomial)

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Encoding of $\alpha = \{x_1,\ldots,x_n\} \longrightarrow V$

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Cost of coding: $||\alpha|| = n\cdot\log(|V|)$

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  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$
- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
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Question:
How much space
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- If $\phi(x_1,\ldots,x_n) = \exists y \,.\, \psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.
  $\rightsquigarrow$ $2 \cdot \log(|G|)$ + SPACE($G \models_{\alpha'} \psi$)

Question:
How much space does it take?

$$\underbrace{2\cdot\log(|G|) + \cdots + 2\cdot\log(|G|)}_{\leq |\phi| \text{ times}} + k\cdot\log(|\alpha|+|G|) \text{ space}$$

Evaluation problem for FO in PSPACE

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$

Output: $G \vDash_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$

  use 4 pointers $\rightsquigarrow$ LOGSPACE

- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \vDash_\alpha \psi$ and $G \vDash_\alpha \psi'$

  $\rightsquigarrow$ MAX(SPACE($G \vDash_\alpha \psi$)), SPACE($G \vDash_\alpha \psi'$)))

- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \vDash_\alpha \psi$

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PSPACE-complete problem: **QBF**

(satisfaction of Quantified Boolean Formulas)

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Theorem: Evaluation for FO is PSPACE-complete (combined c.)

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Polynomial reduction **QBF** $\rightsquigarrow$ FO :

1. Given $\psi \in$ QBF,
let $\psi'(x)$ be the replacement
of each 'p' with 'p=x' in ψ.

2. Note: $\exists x \, \psi'$ holds in a 2-element
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3. Test if $G \models_{\varnothing} \psi'$ for $G=(\{v,v'\},\{\})$

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Combined, Query, and Data complexities

[Vardi, 1982]

Problem: Usual scenario in database

A **database** of size 10^6

A **query** of size 100

Input:

Combined, Query, and Data complexities

[Vardi, 1982]

Problem: Usual scenario in database

A **database** of size 10^6

A **query** of size 100

Input: • **query** +

Combined, Query, and Data comp

Problem: Usual scen

Input: • **query** +

database

Combined, Query, and Data comp

Problem: Usual scen

Input: • **query** +

database

But we don't distinguish this in the analysis:

$$\mathrm{TIME}(2^{|\text{query}|} + |\text{data}|) = \mathrm{TIME}(|\text{query}| + 2^{|\text{data}|})$$

Combined, Query, and Data complexities
[Vardi, 1982]

Query and data play very **different roles.**

Separation of concerns: How the resources grow with respect to

- the size of the data
- the query size

Combined, Query, and Data complexities

Combined complexity: input size is |query| + |data|

Query complexity (|data| fixed): input size is |query|

Data complexity (|query| fixed): input size is |data|

Combined, Query, and Data complexities

Combined complexity: input size is |query| + |data|

Query complexity (|data| fixed): input size is |query|

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$O(2^{|query|} + |data|)$ is
exponential in **combined** complexity
exponential in **query** complexity
linear in **data** complexity

$O(|query| + 2^{|data|})$ is
exponential in **combined** complexity
linear in **query** complexity
exponential in **data** complexity

Question

What is the data, query and combined complexity
for the evaluation problem for FO?

Remember: **data** complexity, input size: |data|

query complexity, input size: |query|

combined complexity, input size: |data| + |query|

$|\phi| \cdot 2 \cdot \log(|G|) + k \cdot \log(|\alpha|+|G|)$ space

Question

What is the data, query and combined complexity
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$|\phi| \cdot 2 \cdot \log(|G|) + k \cdot \log(|\alpha|+|G|)$ space

query

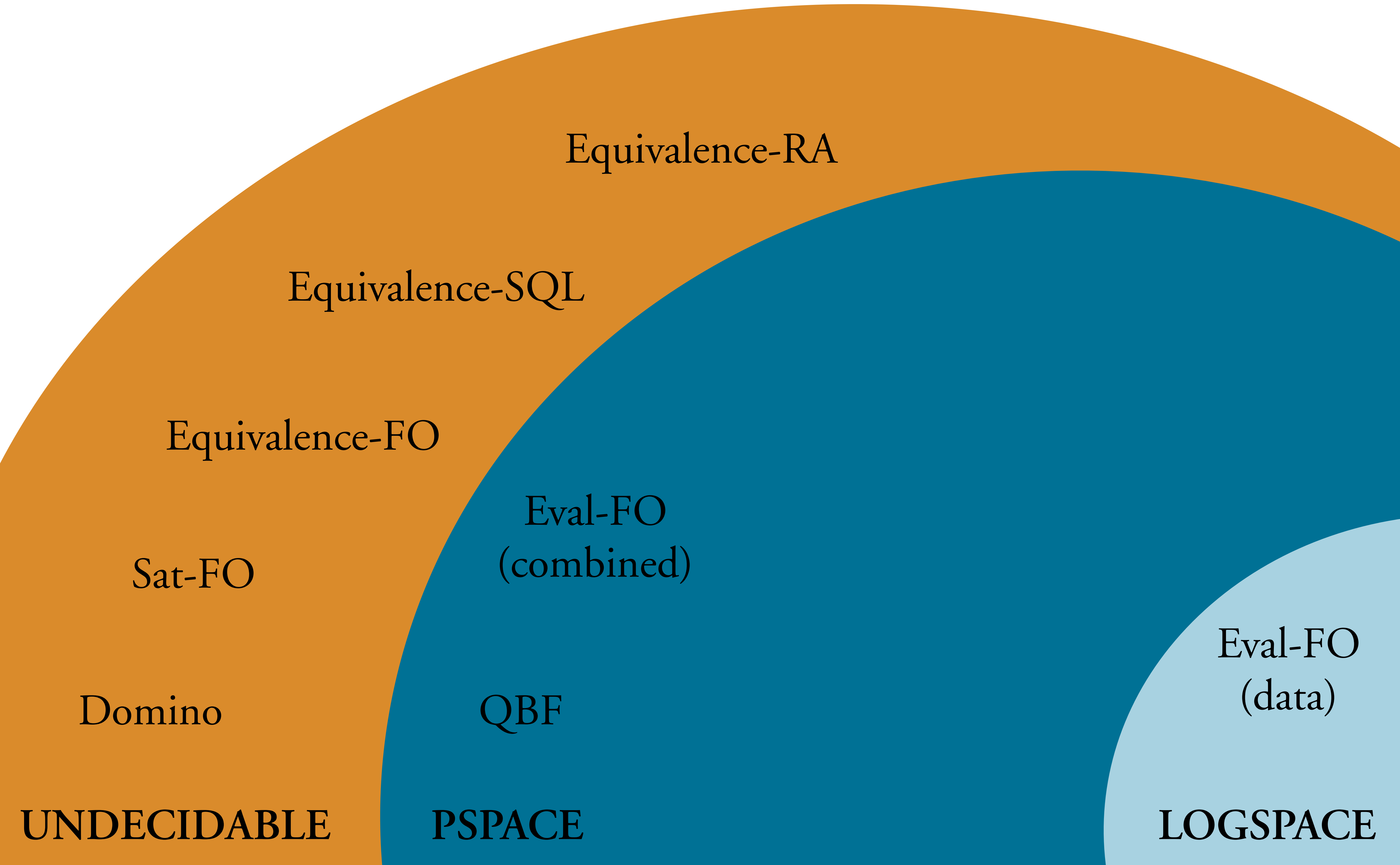
data

O(log(|data|)·|query|) space

PSPACE combined and query complexity

LOGSPACE data complexity

Recap


Equivalence-RA
Equivalence-SQL
Equivalence-FO
Sat-FO
Domino
UNDECIDABLE
Eval-FO
(combined)
QBF
PSPACE
Eval-FO
(data)
LOGSPACE

Trading expressiveness for efficiency

Alternation of quantifiers significantly affects complexity
(recall that evaluation of QBF is PSPACE-complete: $\forall x\ \exists y\ \forall z\ \exists w \ldots \phi$).

What happens if we disallow $\forall$ and $\neg$?

The class NP

$$\text{LOGSPACE} \subseteq \text{PTIME} \subseteq \text{PSPACE} \subseteq \text{EXPTIME}$$

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NP = Problems whose solutions can be witnessed by a ***certificate*** to be guessed and checked in *polynomial time* (e.g. a colouring)

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Examples:

- **3-COLORABILITY**: Given a graph G, can we assign a colour from $\{R,G,B\}$ to each node so that adjacent nodes have always different colours ?
- **SAT**: Given a propositional formula, e.g. $(p \vee \neg q \vee r) \wedge (\neg p \vee s) \wedge (\neg s \vee \neg p)$, can we assign a truth value to each variable so that the formula becomes true ?
- **MONEY-CHANGE**: Given an amount of money A and a set of coins $\{B_1, \ldots, B_n\}$, can we find a subset $S \subseteq \{B_1, \ldots, B_n\}$ such that $\sum S = A$?

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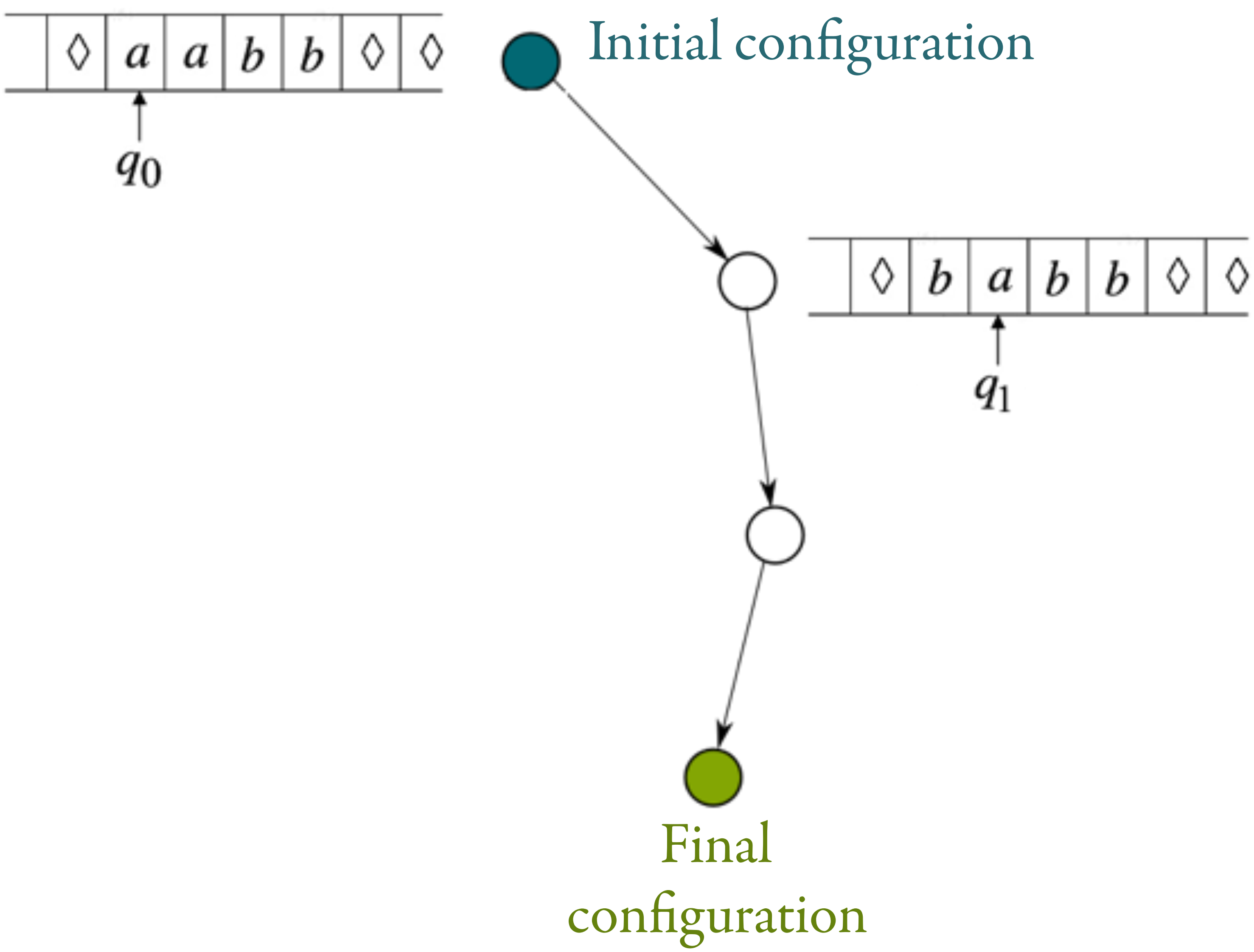

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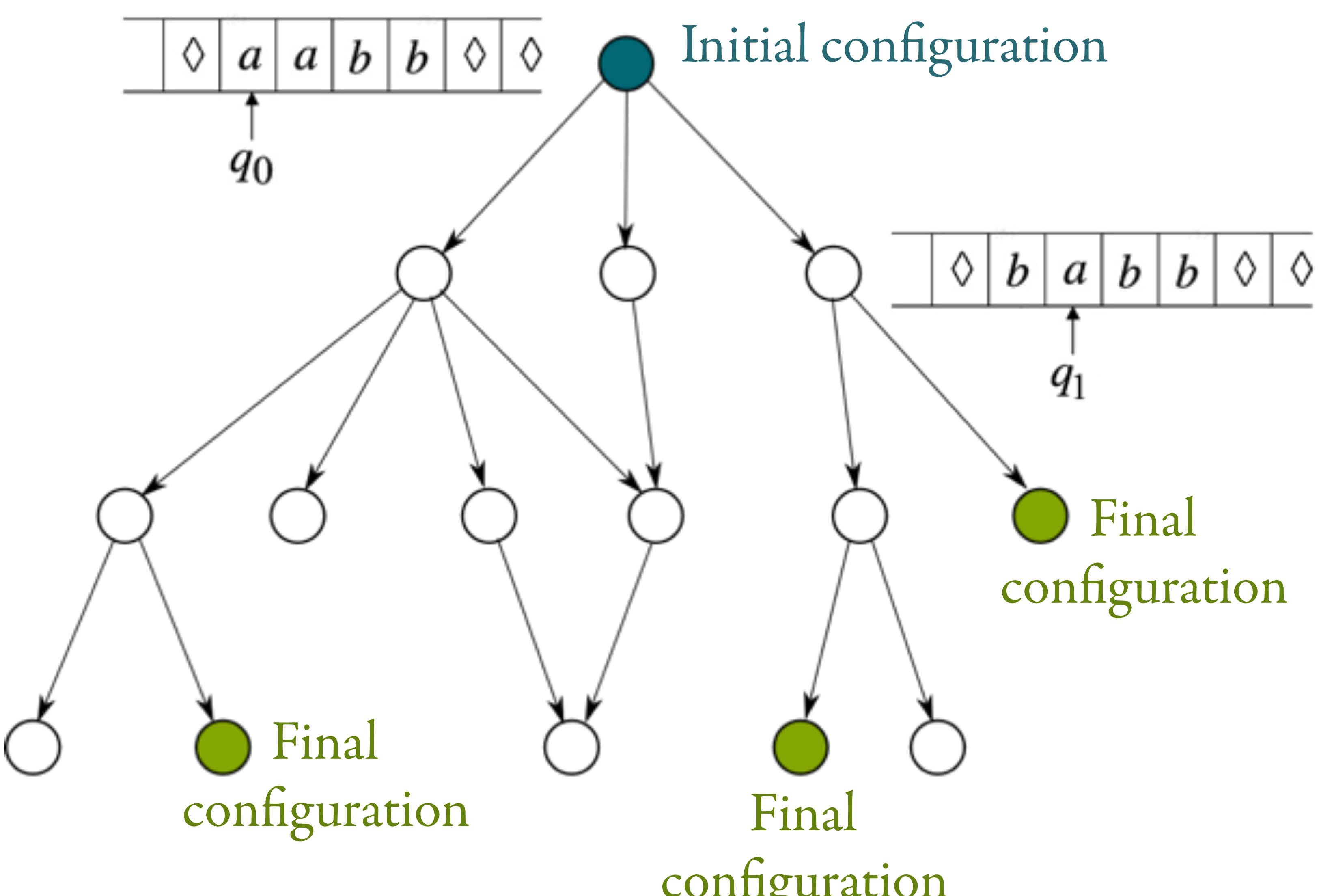

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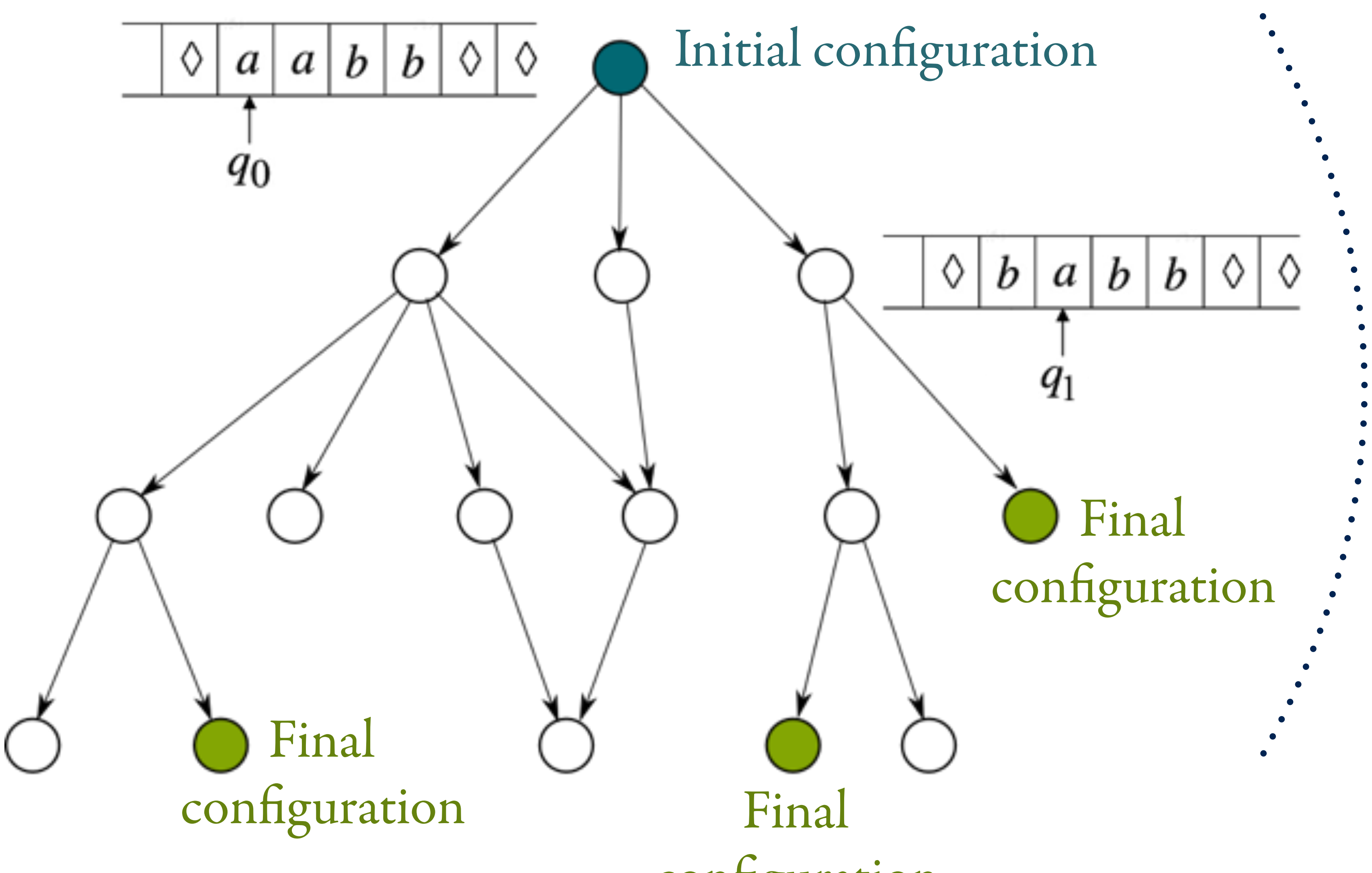

Non-deterministic transitions

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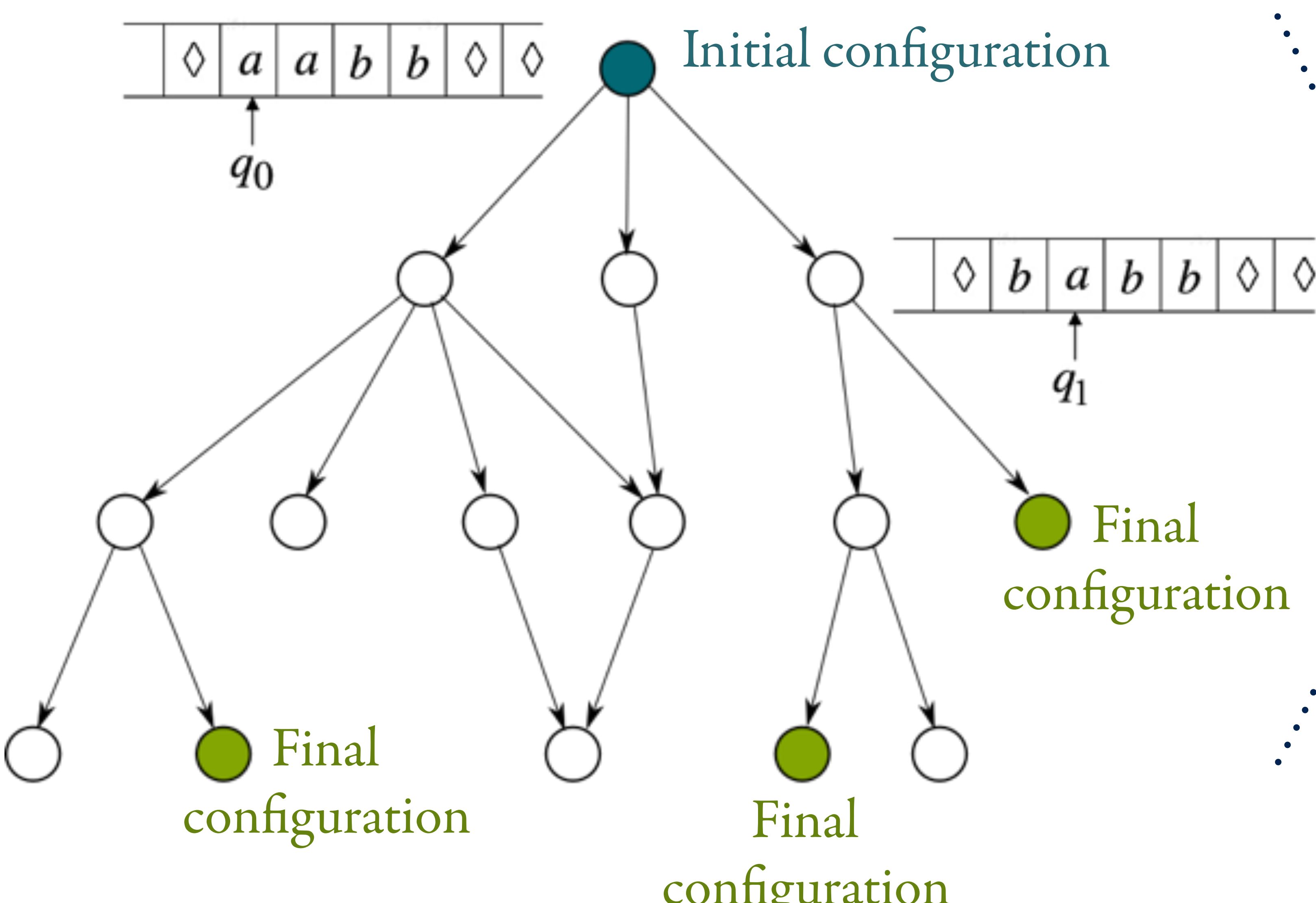

Non-deterministic transitions

Many paths, each has length bounded by a **polynomial**

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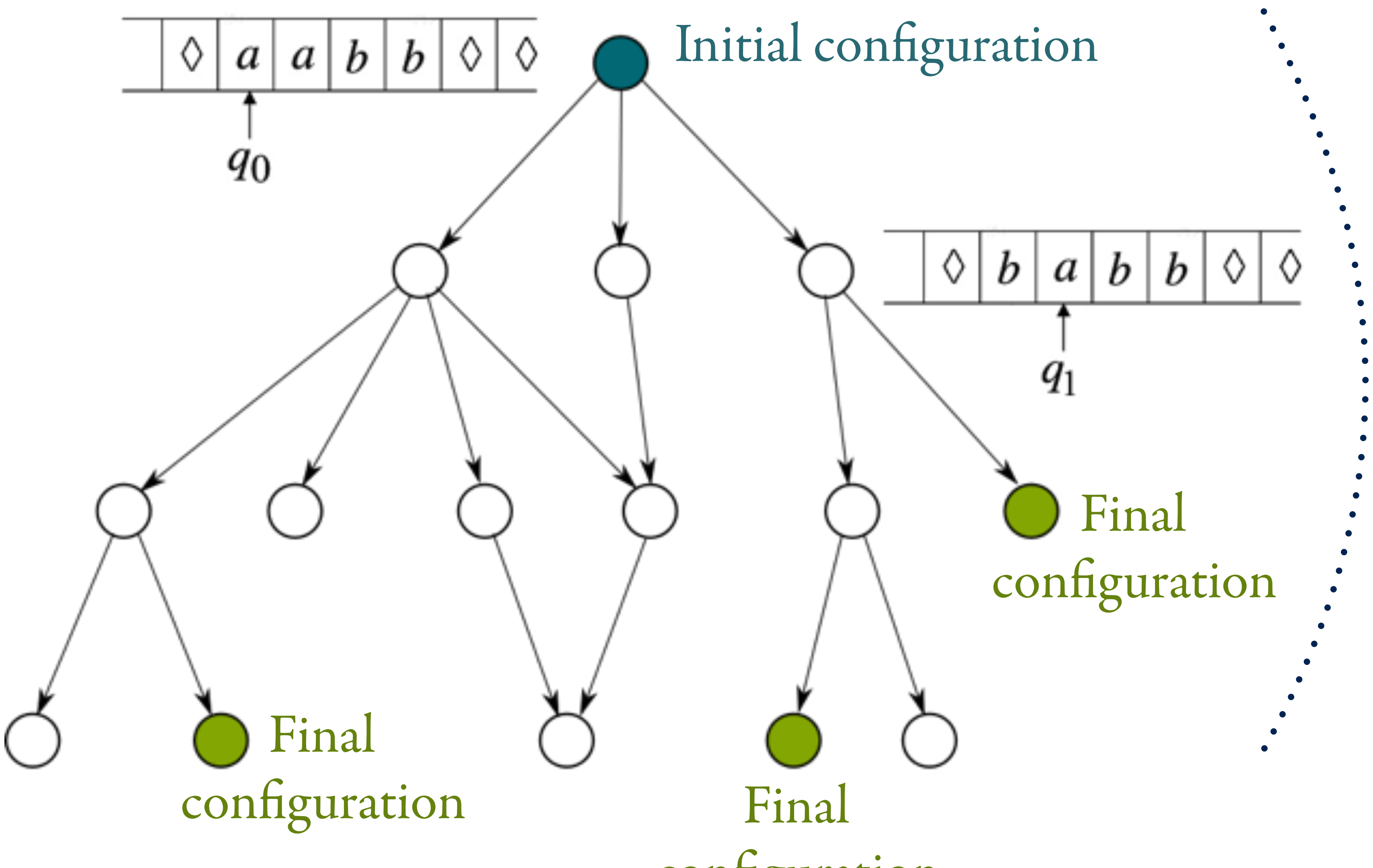

Non-deterministic transitions

Many paths, each has length bounded by a **polynomial**

A solution exists if there is at least a **successful** path.

Question

Consider: **Positive FO = FO without ∀,¬**

E.g. $\phi = \exists x \, \exists y \, \exists z \, . \, \big(E(x, y) \lor E(y, z)\big) \land \big(y{=}z \lor E(x, z)\big)$

What is the complexity of evaluating Positive FO on graphs ?

Question

Consider: **Positive FO = FO without ∀,¬**

E.g. $\phi = \exists x\, \exists y\, \exists z\, .\, \big(E(x, y) \vee E(y, z)\big) \wedge \big(y{=}z \vee E(x, z)\big)$

What is the complexity of evaluating Positive FO on graphs ?

Solution

This is in NP: Given ϕ and $G{=}(V, E)$
it suffices to guess a binding $\alpha : \{ x, y, z, \dots \} \to V$
and then verify that the formula holds.

Conjunctive Queries

Def.

CQ = FO without $\forall, \neg, \vee$

Eg: $\phi(x, y) = \exists z \,.\, (\mathrm{Parent}(x, z) \wedge \mathrm{Parent}(z, y))$

Usual notation: "Grandparent(X,Y) : – Parent(X,Z), Parent(Z,Y)"

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Normal form: " $\exists\, x_1, \ldots, x_n \,.\, \phi(x_1, \ldots, x_n)$ "

quantifier-free and no equalities!

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It corresponds to positive "SELECT-FROM-WHERE" SQL queries

```
Select ...
From ...
Where Z
```

no negation or disjunction

It corresponds to "π-σ-×" RA queries

$\pi_X(\sigma_Z(R_1 \times \cdots \times R_n))$

no negation

Bibliography

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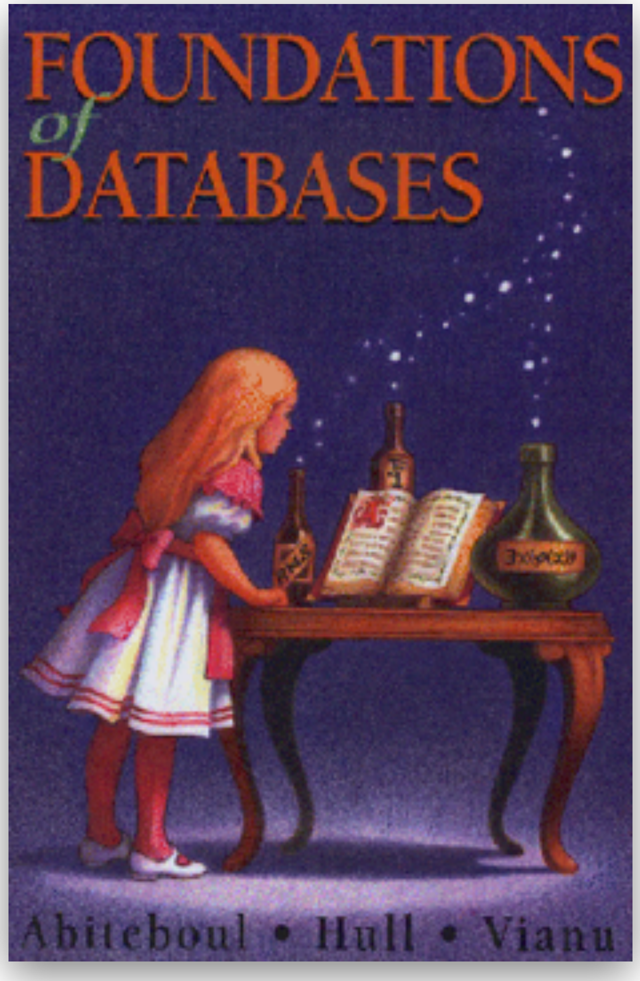
(freely available at http://webdam.inria.fr/Alice/)



Chapters 1, 2, 3