day 5

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Logical foundations of databases

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Recap

- **Acyclic Conjunctive Queries**
- **Join Trees**
- **Evaluation of ACQ** (LOGCFL-complete)
- **Ears, GYO algorithm for testing acyclicity**
- **Tree decomposition, tree-width of CQ**
- **Evaluation of bounded tree-width CQs** (LOGCFL-complete)
- **Bounded variable fragment of FO, evaluation in PTIME**
- Acyclic Conjunctive Queries

Ehrenfeucht-Fraïssé games

Duplicator

Spoiler

They play for n rounds on the board (S_1, S_2).

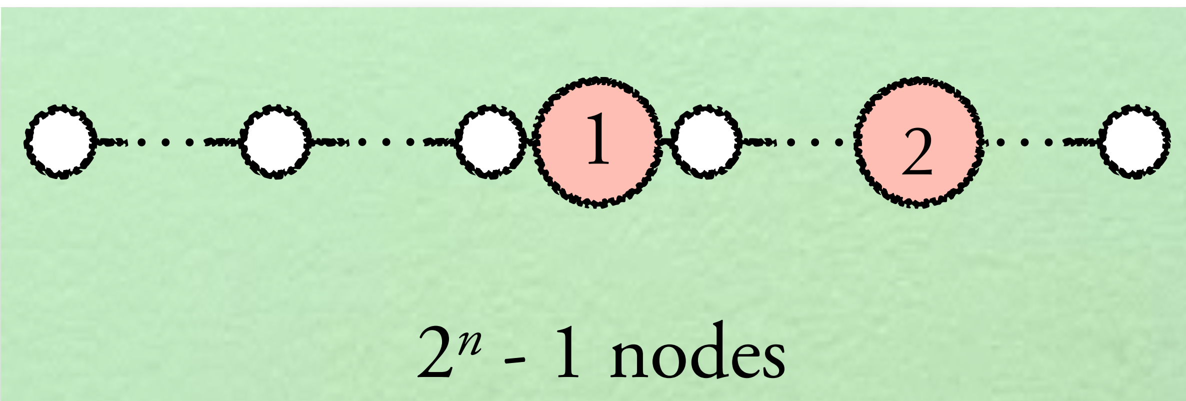
At each round i : Spoiler chooses a node x_i from S_1 (resp. y_i from S_2)

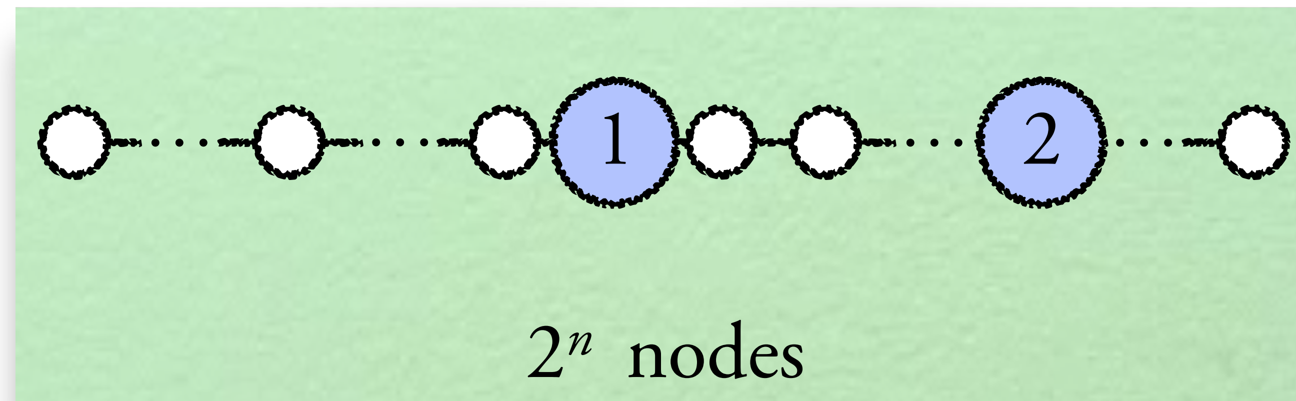
Duplicator answers with a node y_i from S_2 (resp. x_i from S_1)
trying to maintain an isomorphism between $S_1 | \{x_i\}_i$ and $S_2 | \{y_i\}_i$

Ehrenfeucht-Fraïssé games

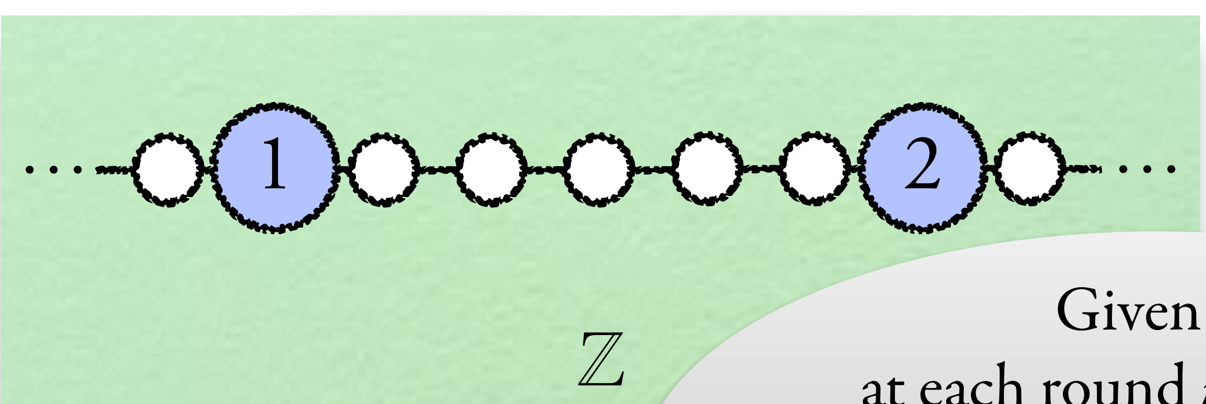
On non-isomorphic *finite* structures, Spoiler wins eventually... **Why?**

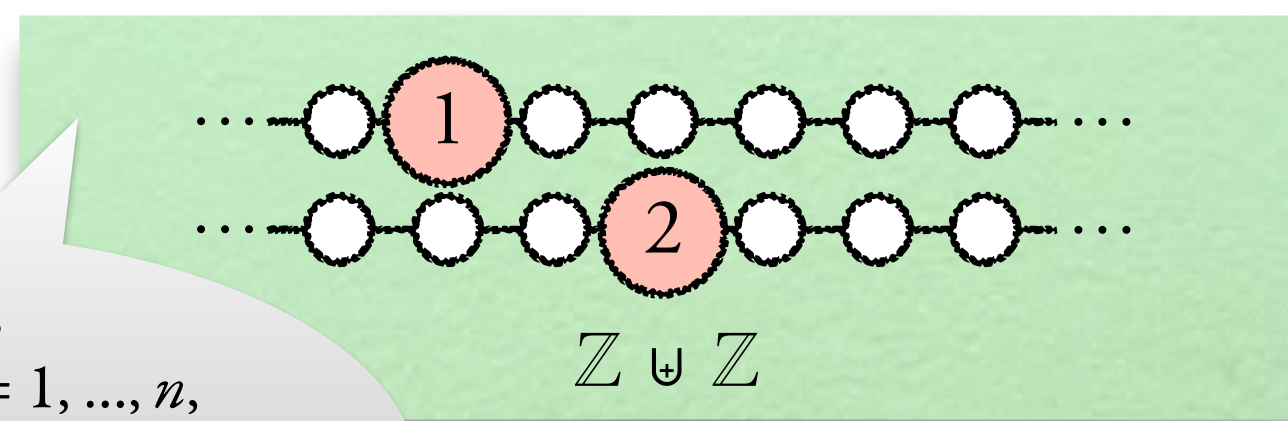
...and he often wins very quickly:





But there are non-isomorphic *infinite* structures
where Duplicator can survive for *arbitrarily many rounds* (not necessarily forever!)





Given n,
at each round $i = 1, \ldots, n$,
pairs of marked nodes in S_1 and S_2
must be either at *equal distance*
or *at distance* $\geq 2^{n-i}$

Ehrenfeucht-Fraïssé games

Theorem. S_1 and S_2 are n - *equivalent* [Fraïssé '50, Ehrenfeucht '60]

iff Duplicator has a strategy to survive n rounds in the EF game on S_1 and S_2.

Proof ideas for the if-direction (from Duplicator's winning strategy to n - equivalence)

Consider ϕ with quantifier rank n.

Suppose $S_1 \vDash \phi$ and Duplicator survives n rounds on S_1, S_2.

We need to prove that $S_2 \vDash \phi$.

A new game to evaluate formulas....

The semantics game

Assume w.l.o.g. that ϕ is in **negation normal form.**

push negations inside:

$\neg\forall\phi \rightsquigarrow \exists\neg\phi$

$\neg\exists\phi \rightsquigarrow \forall\neg\phi$

$\neg(\phi \wedge \psi) \rightsquigarrow \neg\phi \vee \neg\psi$

...

Whether $S \vDash \phi$ can be decided by a **new game** between two players, True and False:

- $\phi = E(x,y)$ → True wins if nodes marked x and y are connected by an edge, otherwise he loses
- $\phi = \exists\, x\ \phi'(x)$ → True moves by marking a node x in S, the game continues with ϕ'
- $\phi = \forall\, y\ \phi'(y)$ → False moves by marking a node y in S, the game continues with ϕ'
- $\phi = \phi_1 \vee \phi_2$ → True moves by choosing ϕ_1 or ϕ_2, the game continues with what he chose
- $\phi = \phi_1 \wedge \phi_2$ → False moves by choosing ϕ_1 or ϕ_2, the game continues with what he chose
- ...

Lemma. $S \vDash \phi$ iff True wins the semantics game.

Ehrenfeucht-Fraïssé games

Theorem. S_1 and S_2 are *n - equivalent* [Fraïssé '50, Ehrenfeucht '60]

iff Duplicator has a strategy to survive n rounds in the EF game on S_1 and S_2.

Proof ideas for the if-direction (from Duplicator's winning strategy to n - equivalence)

True wins the game on S_1

Consider ϕ with quantifier rank n.

Suppose $S_1 \models \phi$ and Duplicator survives n rounds on S_1, S_2.

We need to prove that $S_2 \models \phi$.

True wins the game on S_2

Turn winning strategy for True in S_1 into winning strategy for True in S_2 ….

Ehrenfeucht-Fraïssé games

Theorem. S_1 and S_2 are n-*equivalent* [Fraïssé '50, Ehrenfeucht '60]

iff Duplicator has a strategy to survive n rounds in the EF game on S_1 and S_2.

Proof ideas for the if-direction (from Duplicator's winning strategy to n-equivalence)

True wins the game on S_1

Consider ϕ with quantifier rank n.

Suppose $S_1 \models \phi$ and Duplicator survives n rounds on S_1, S_2.

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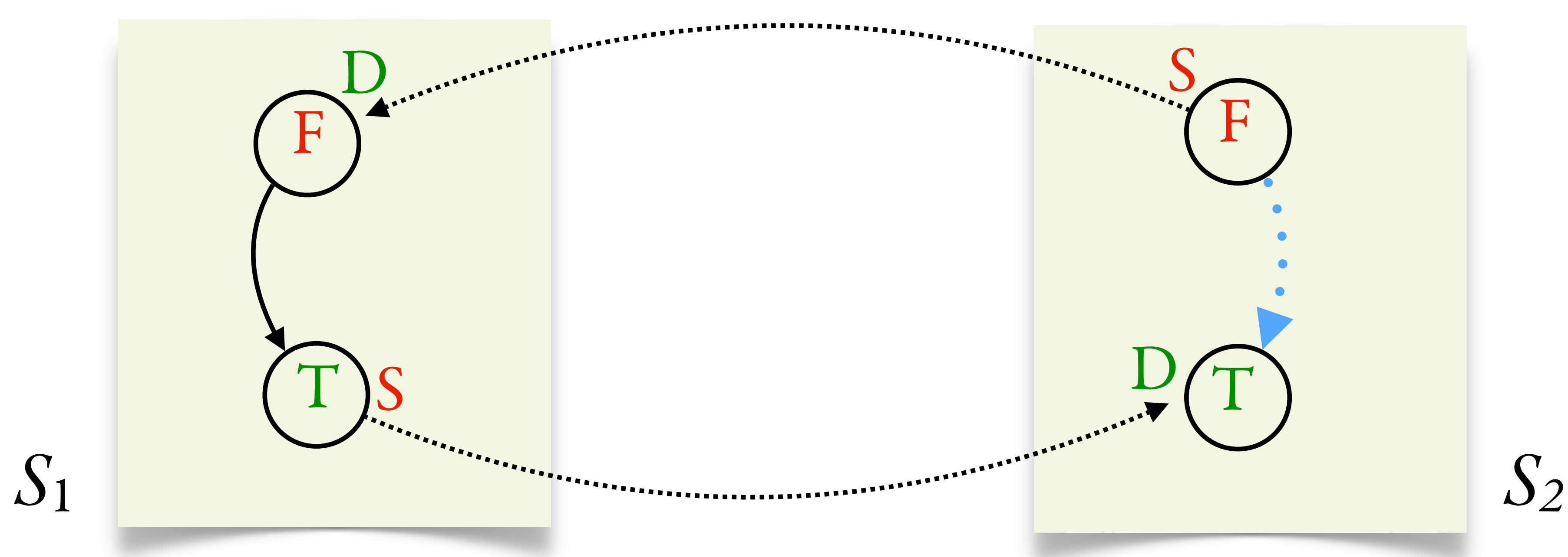


Definability in FO

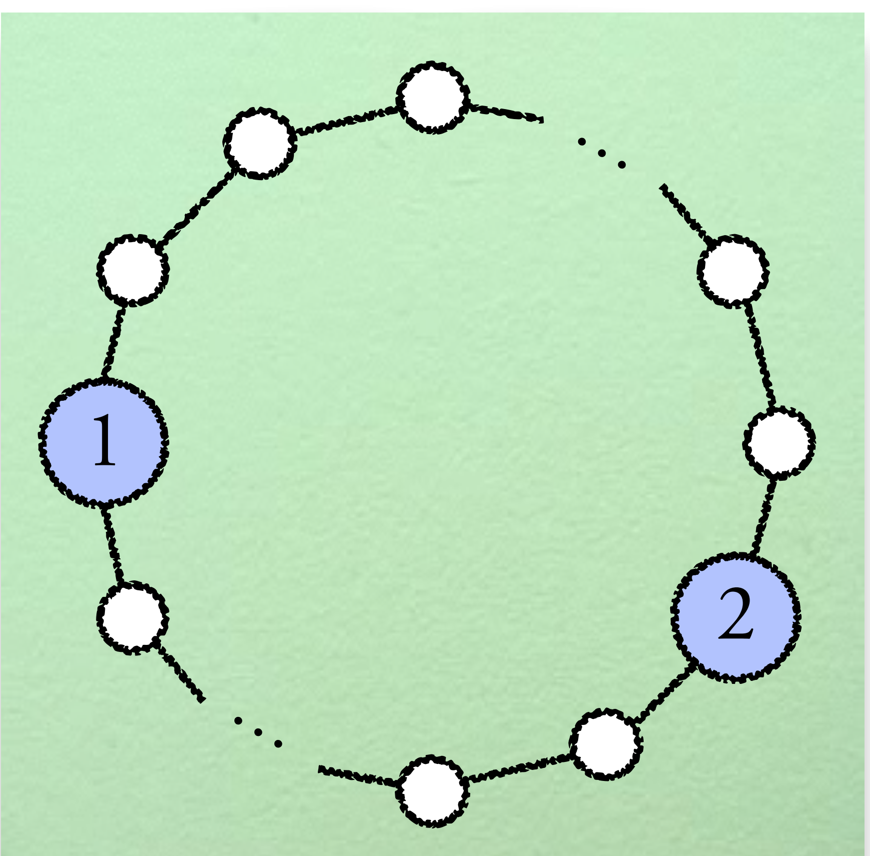
Theorem. S_1 and S_2 are *n-equivalent* [Fraïssé '50, Ehrenfeucht '60]

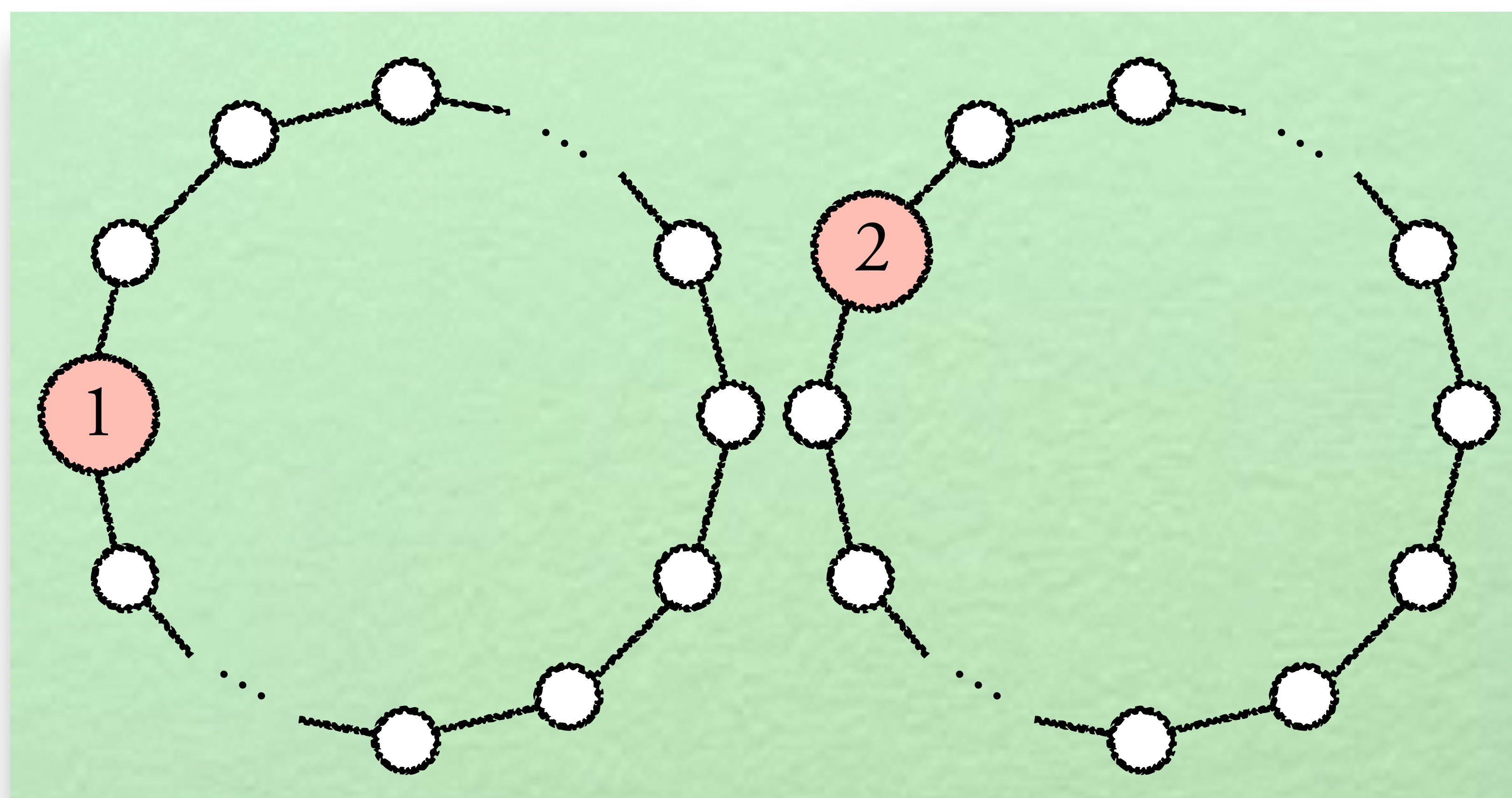
iff Duplicator has a strategy to survive n rounds in the EF game on S_1 and S_2.

Corollary. A property P is *not definable in FO*

iff $\forall n \;\; \exists S_1 \in P \;\; \exists S_2 \notin P$ Duplicator can survive n rounds on S_1 and S_2.

Example: $P = \{$ connected graphs $\}$. Given n, take $S_1 \in P$ large enough and $S_2 = S_1 \uplus S_1 \notin P$

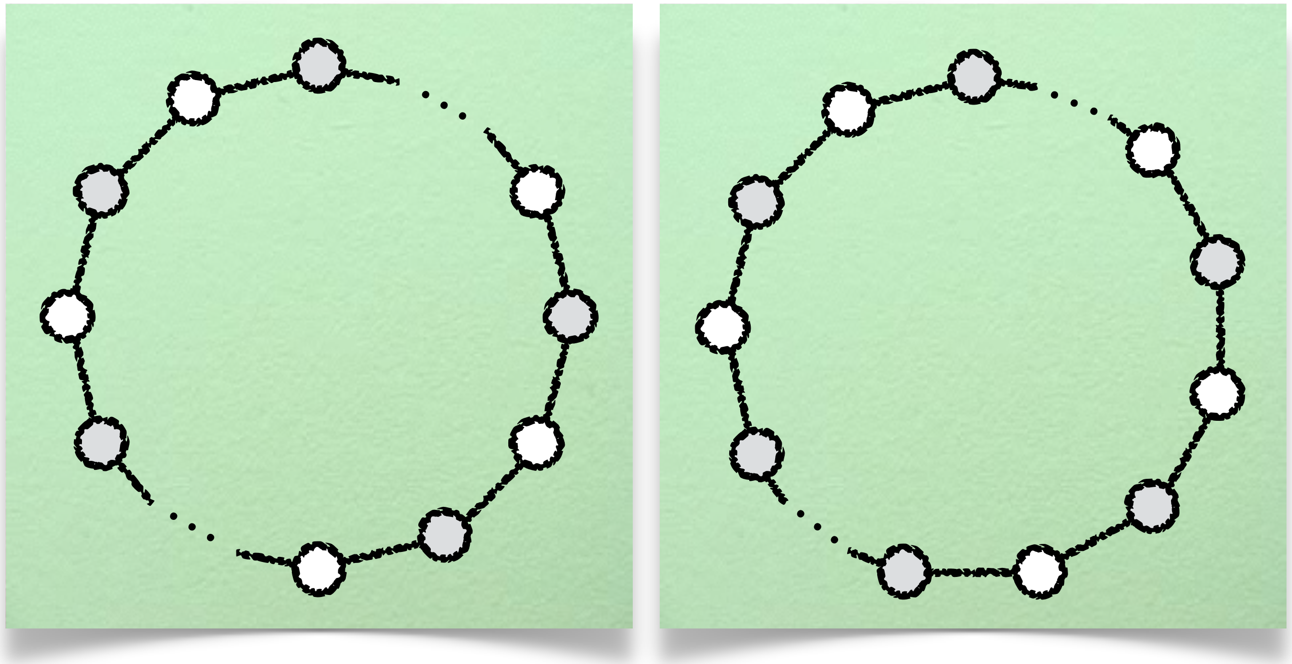




Ehrenfeucht-Fraïssé games

Several properties can be proved to be *not FO-definable*:

- connectivity (previous slide)
- even / odd size **Your turn now!** ...given n, take S_1 = large even structure
  S_2 = large odd structure...
- 2-colorability Given n, take S_1 = large even cycle S_2 = large odd cycle



- finiteness
- acyclicity

...

0-1 Law

A different perspective: a coarser view on expressiveness…

What percentage of graphs verify a given FO sentence?

0-1 Law

$\mu_n(\mathbf{P})$ = "probability that property **P** holds in a **random** graph with ***n*** nodes"

Uniform distribution

(each pair of nodes has an edge with probability ½)

C_n = { graphs with n nodes }

$$\mu_n(\mathbf{P}) = \frac{|\{G \in C_n \mid G \models \mathbf{P}\}|}{|C_n|}$$

$$|C_n| = 2^{n^2}$$

E.g. for **P** = "the graph is complete"

$$\mu_3(\mathbf{P}) = \frac{1}{|C_3|} = \frac{1}{2^{3^2}}$$

$$\mu_\infty(\mathbf{P}) = \lim_{n \to \infty} \mu_n(\mathbf{P})$$

0-1 Law

Theorem. [Glebskii et al. '69, Fagin '76]

For every *FO sentence* ϕ, $\mu_\infty(\phi)$ is either 0 or 1 .

Examples:

- ϕ = "there is a triangle" $\mu_3(\phi) = {}^1/_{|C_3|}$ $\mu_{3n}(\phi) \geq 1 - (1 - {}^1/_{|C_3|})^n \to 1$
- ϕ_H = "there is an occurrence of *H as induced sub-graph*" $\mu_\infty(\phi_H) = 1$
- ϕ = "there no 5-clique" $\mu_\infty(\phi) = 0$
- ϕ = "even number of edges"
- ϕ = "even number of nodes"

Your turn!

$\mu_\infty(\phi) = {}^1/_2$

$\mu_\infty(\phi)$ not even defined

- ϕ = "more edges than nodes" $\mu_\infty(\phi) = 1$

(yet not FO-definable!)

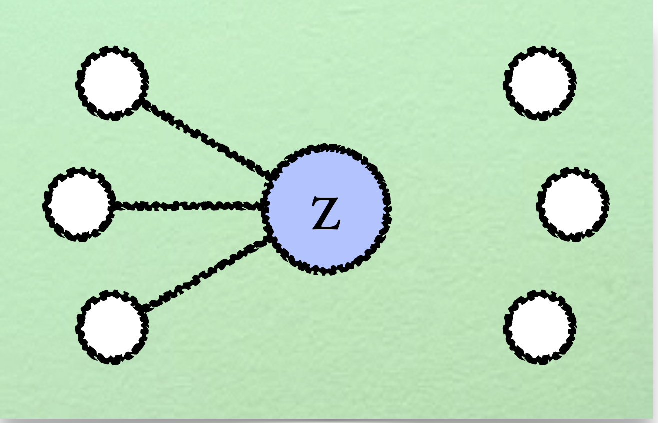
0-1 Law

For every *FO sentence* ϕ, $\mu_\infty(\phi)$ is either 0 or 1.

Let k = quantifier rank of ϕ

$$\delta_k = \forall x_1, \ldots, x_k \, \forall y_1, \ldots, y_k \, \exists z \; \bigwedge_{i,j} x_i \neq y_j \wedge E(x_i, z) \wedge \neg E(y_j, z)$$

(Extension Formula/Axiom)



Fact 1: If $G \models \delta_k \wedge H \models \delta_k$ then Duplicator survives k rounds on G, H

Fact 2: $\mu_\infty(\delta_k) = 1$ (δ_k is almost surely true)

2 cases

a) There is G $G \models \delta_k \wedge \phi$ $\Rightarrow$ (by Fact 1) $\forall H$: If $H \models \delta_k$ then $H \models \phi$

Thus, $\mu_\infty(\delta_k) \leq \mu_\infty(\phi)$

$\Rightarrow$ (by Fact 2) $\mu_\infty(\delta_k) = 1$, hence $\mu_\infty(\phi) = 1$

b) There is no $G \models \delta_k \wedge \phi$ $\Rightarrow$ (by Fact 2) there is $G \models \delta_k$,

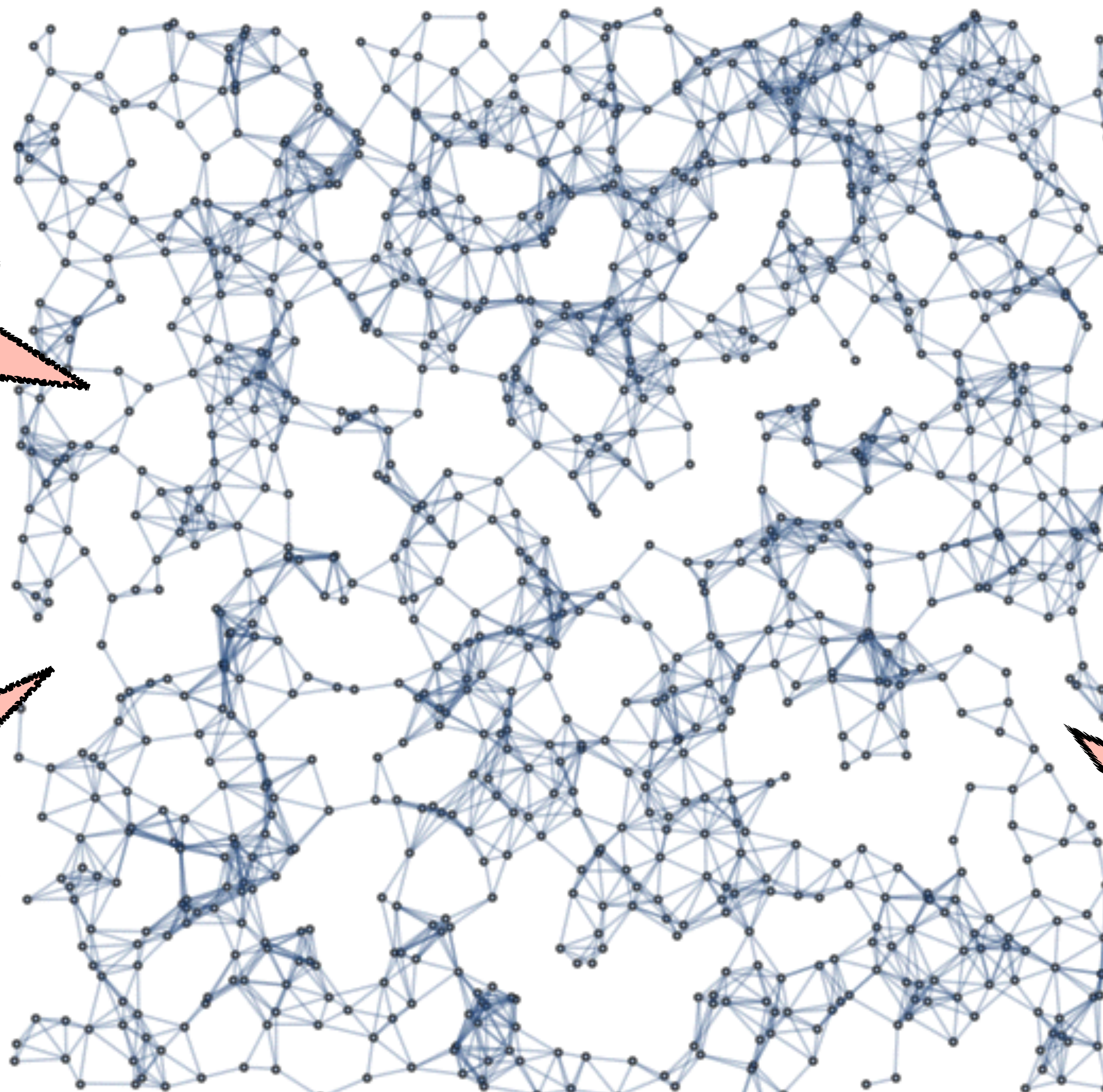
$\Rightarrow$ $G \models \delta_k \wedge \neg\phi$ $\Rightarrow$ (by case a) $\mu_\infty(\neg\phi) = 1$

0-1 Law

For every *FO sentence* ϕ, $\mu_\infty(\phi)$ is either 0 or 1, **and this depends on whether RADO $\models \phi$**

each pair of nodes i, j is connected if i-th bit of j is 1

RADO =

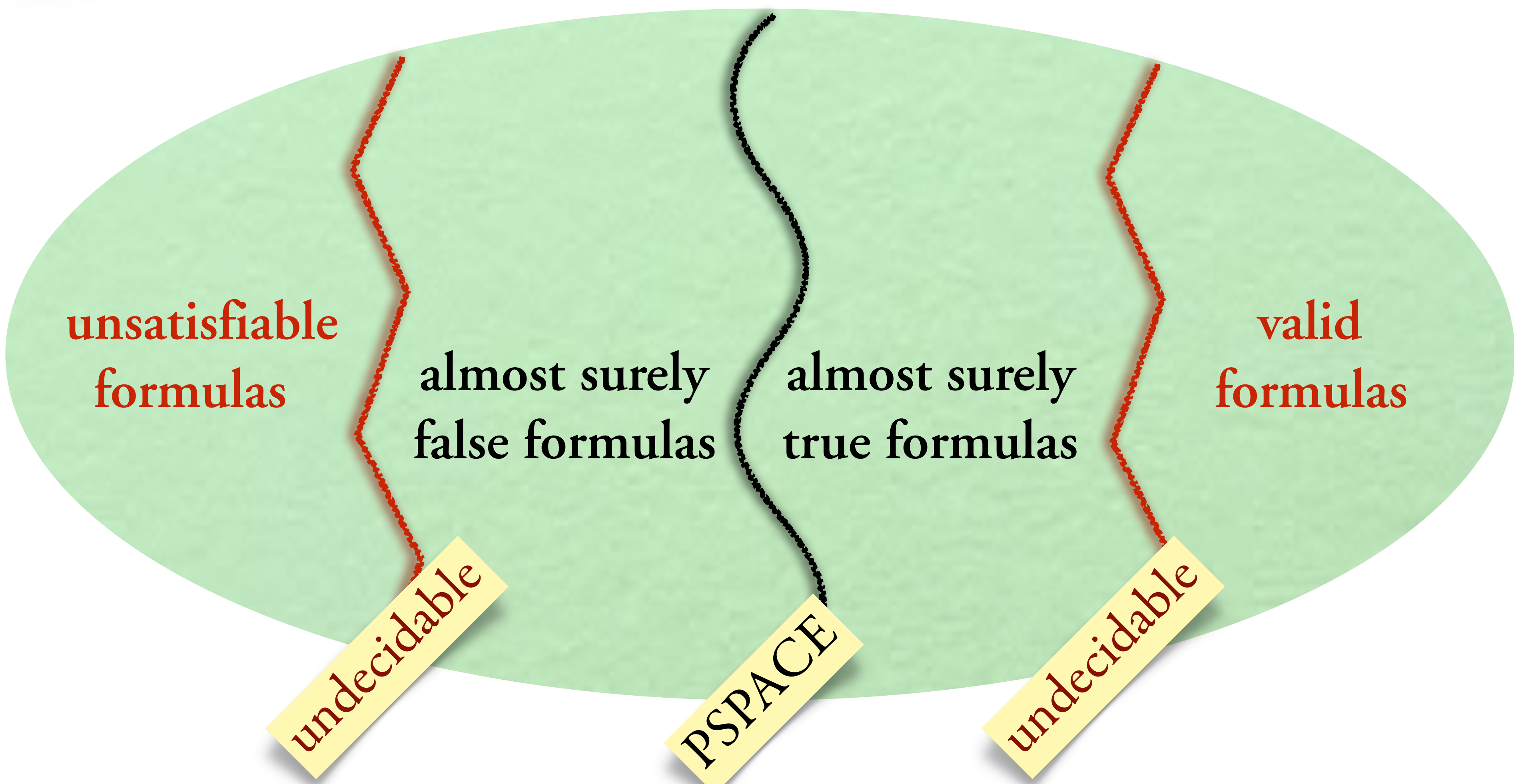


each pair of nodes i, j is connected with probability 1/2

the unique graph that satisfies δ_k for all k

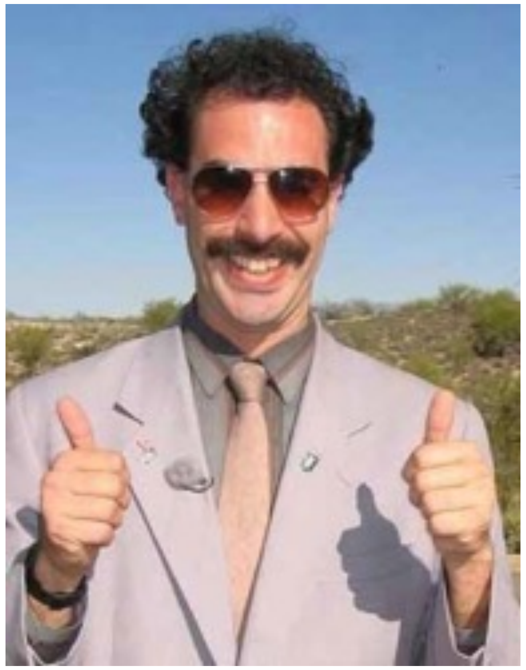
0-1 Law

Theorem. The problem of deciding whether an FO sentence is *almost surely true* $(\mu_\infty = 1)$ is PSPACE-complete. [Grandjean '83]



Query evaluation on large databases:

Don't bother evaluating an FO query,
it's either *almost surely true* or *almost surely false*!



0-1 Law

Does the 0-1 Law apply to real-life databases?

Not quite: database *constraints* easily spoil Extension Axiom.

Consider:

- functional constraint $\forall\, x, x', y, y' \ \big(E(x,y) \wedge E(x,y') \Rightarrow y = y' \big) \wedge$
  $\big(E(x,y) \wedge E(x',y) \Rightarrow x = x' \big)$ (E is a permutation)
- FO query $\phi = \neg \exists\, x \ E(x,x)$

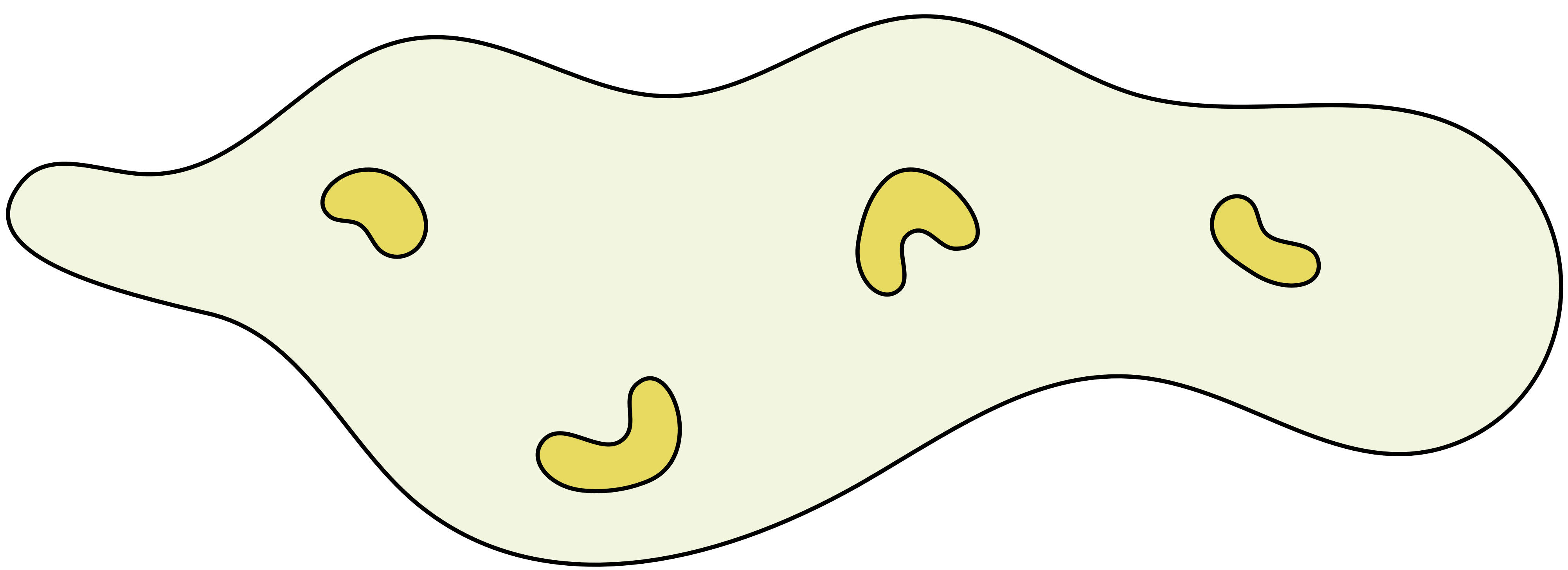
Probability that a permutation E satisfies $\phi = {}^{!n}/_{n!} \to e^{-1} = 0.3679\ldots$

0-1 Law only applies to **unconstrained** databases…

Another technique: Locality

Idea: First order logic can only express "local" properties

Local = properties of nodes which are close to one another



Hanf locality

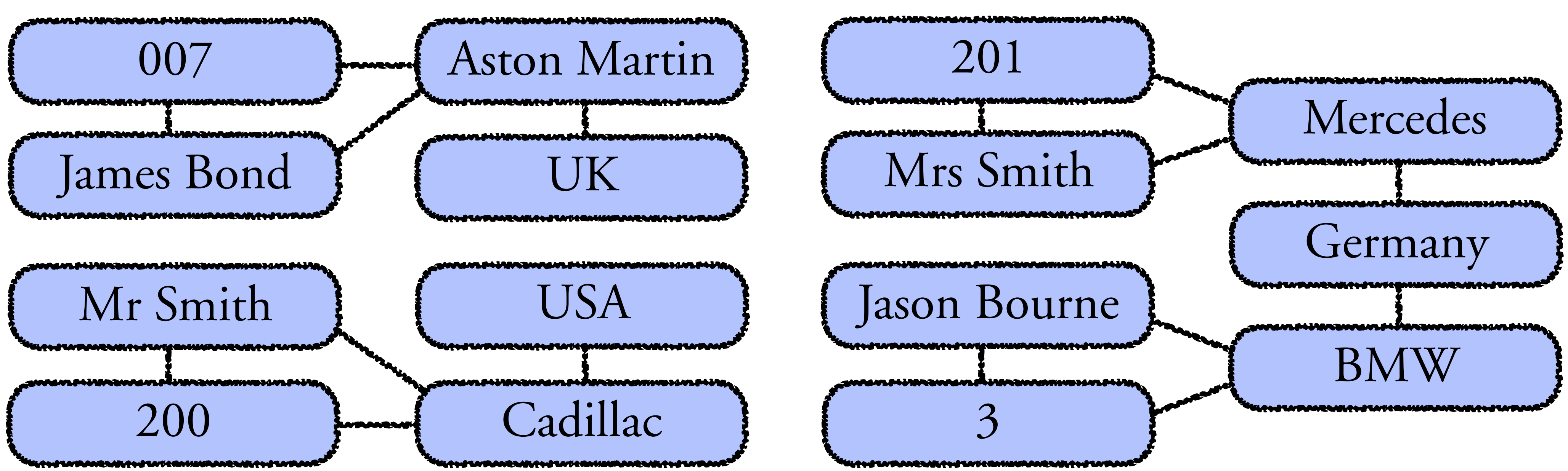
Definition. The **Gaifman graph** of a structure $S = (V, R_1, \ldots, R_m)$ is the **undirected** graph

$$G_S = (V, E) \quad \text{where} \quad E = \{ (u, v) \mid \exists (\ldots, u, \ldots, v, \ldots) \in R_i \text{ for some } i \}$$

Agent	Name	Drives
007	James Bond	Astor
200	Mr Smith	Cadi
201	Mrs Smith	Mercedes
3	Jason Bourne	BMW

Car	Country
[illegible]	UK
[illegible]	USA
Mercedes	Germany
BMW	Germany

The Gaifman graph of a graph G is the underlying undirected graph.



Hanf locality

- $\mathrm{dist}\,(u, v)$ = **distance** between u and v in the Gaifman graph
- $S\,[u, r]$ = sub-structure induced by $\{\, v \mid \mathrm{dist}\,(u, v) \leq r \,\}$ = **ball** around u of radius r

Agent	Name	Drives
007	James Bond	Aston Martin
200	Mr Smith	Cadillac
201	Mrs Smith	Mercedes u
3	Jason Bourne	BMW

Car	Country
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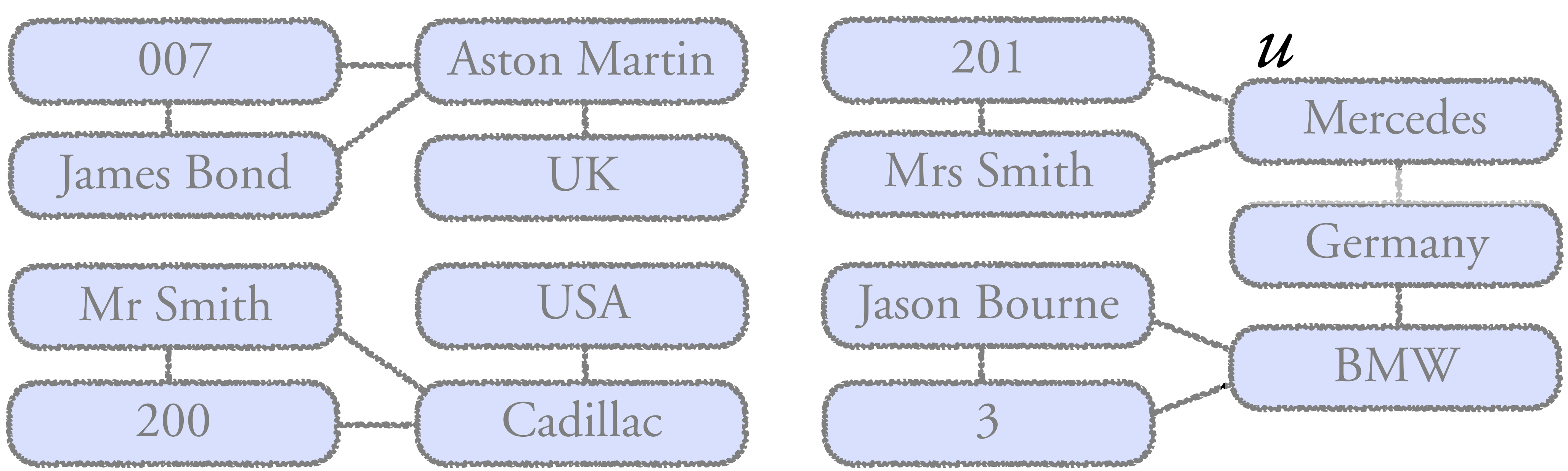
u Mercedes	Germany
BMW	Germany



Hanf locality

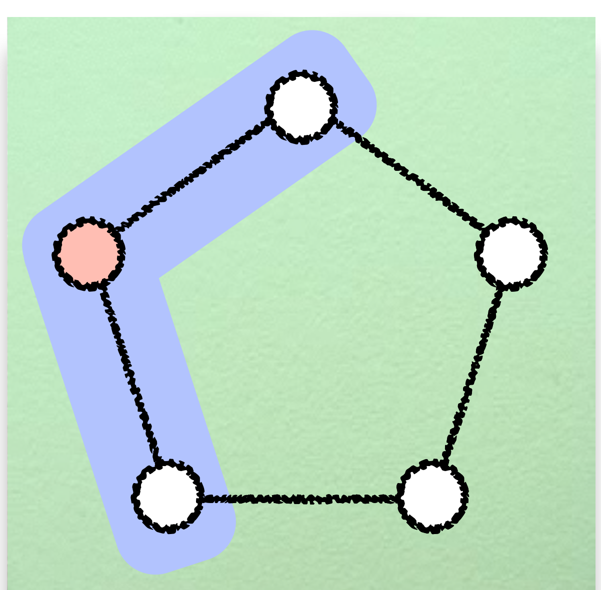
Definition. Two structures S_1 and S_2 are **Hanf (r, t) - equivalent**

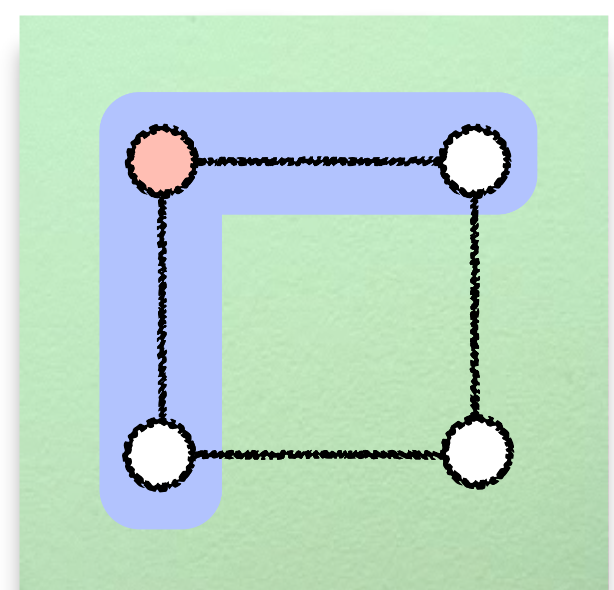
iff for each structure B, the two numbers

$$\#u \text{ s.t. } S_1[u,r] \cong B \qquad \#v \text{ s.t. } S_2[v,r] \cong B$$

are *either the same* or *both* $\geq t$.

Example. S_1, S_2 are Hanf$(1, 1)$ - equivalent iff they have the *same balls* of radius 1





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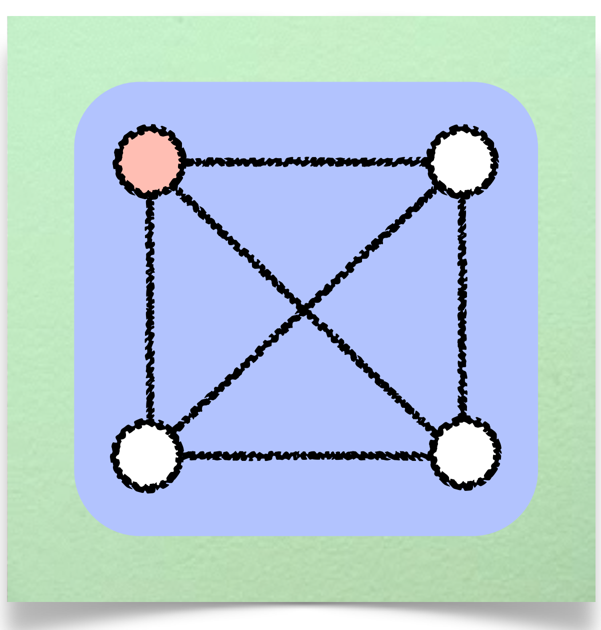
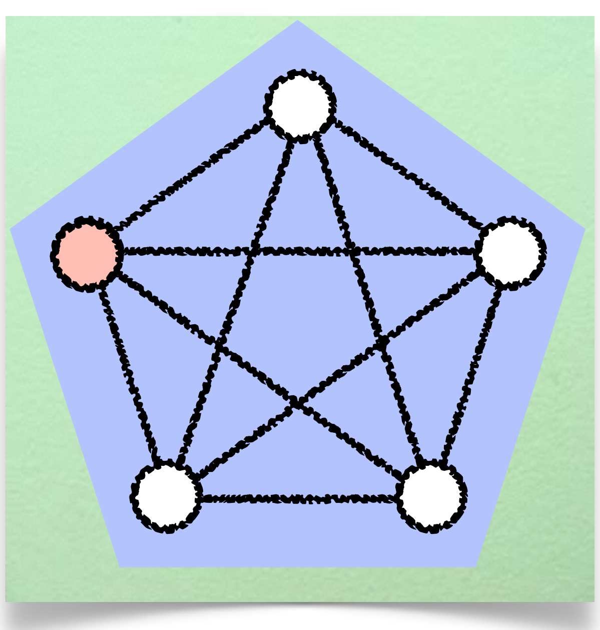
are *either the same* or *both* $\geq t$.

Example. K_n, K_{n+1} are **not** Hanf$(1, 1)$-equivalent



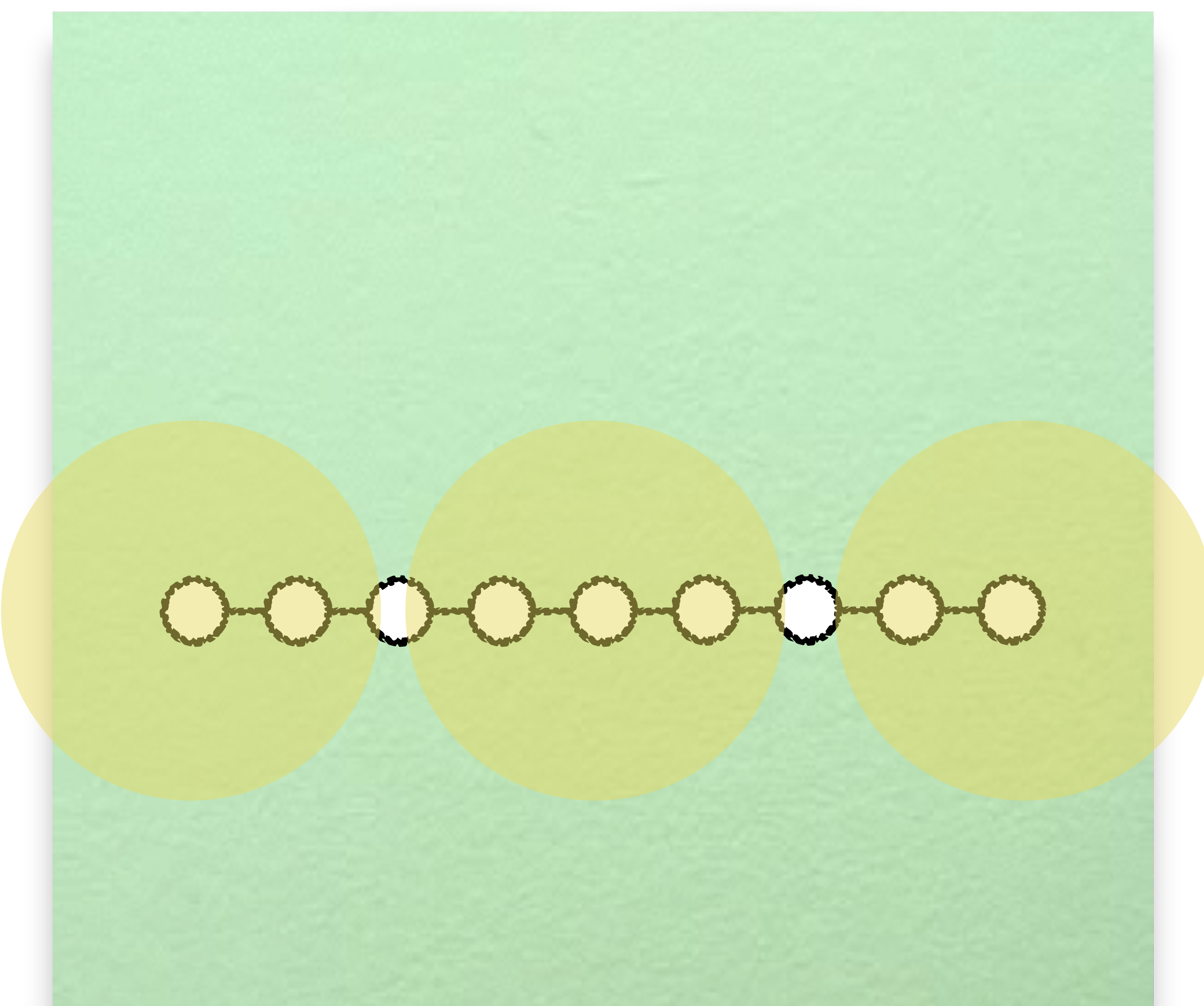


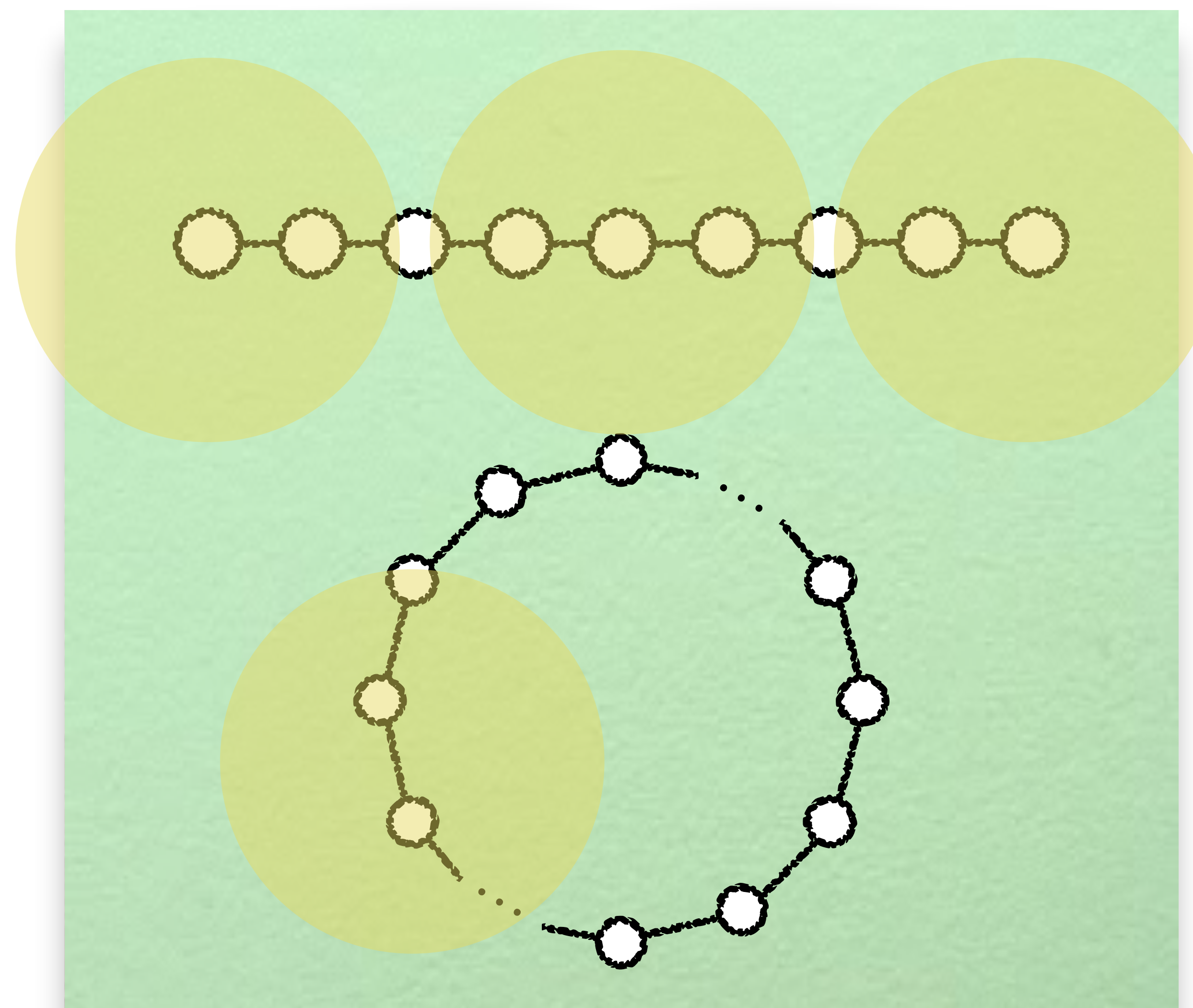
Hanf locality

Theorem. If S_1, S_2 are **Hanf(r, t) - equivalent**, with $r = 3^n$ and $t = n$
then S_1, S_2 are **n - equivalent** (they satisfy the same sentences with quantifier rank n)

[Hanf '60]

Exercise: prove that *acyclicity* is not FO-definable (on finite structures)





Hanf locality

Theorem. S_1, S_2 are n-equivalent (they satisfy the same sentences with quantifier rank n)
whenever S_1, S_2 are Hanf(r,t)-equivalent, with $r = 3^n$ and $t = n$.
[Hanf '60]

Exercise: prove that testing whether a binary tree is *complete* is not FO-definable

Hanf locality

Theorem. S_1, S_2 are n-equivalent (they satisfy the same sentences with quantifier rank n) whenever S_1, S_2 are Hanf(r,t)-equivalent, with $r = 3^n$ and $t = n$.

[Hanf '60]

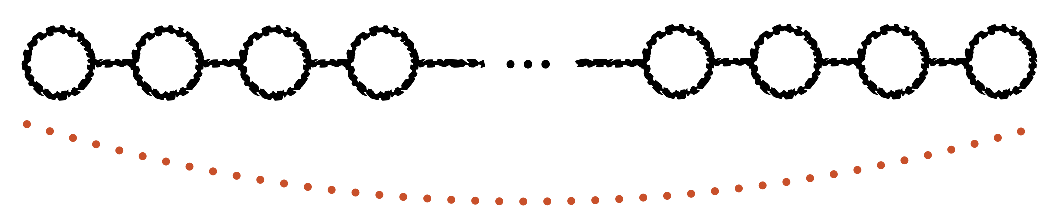
Why so **BIG**?

Remember $\phi_k(x,y)$ = "there is a path of length 2^k from x to y"

$$\phi_0(x, y) = E(x, y), \text{ and}$$
$$\phi_k(x,y) = \exists z\, (\, \phi_{k-1}(x, z) \wedge \phi_{k-1}(z, y)\,)$$

$$qr(\phi_k) = k$$

$2{\cdot}2^n{+}1$



$2{\cdot}2^n$

Not (n+2)-equivalent yet they have the same $2^n{-}1$ balls.

Gaifman locality

What about queries?

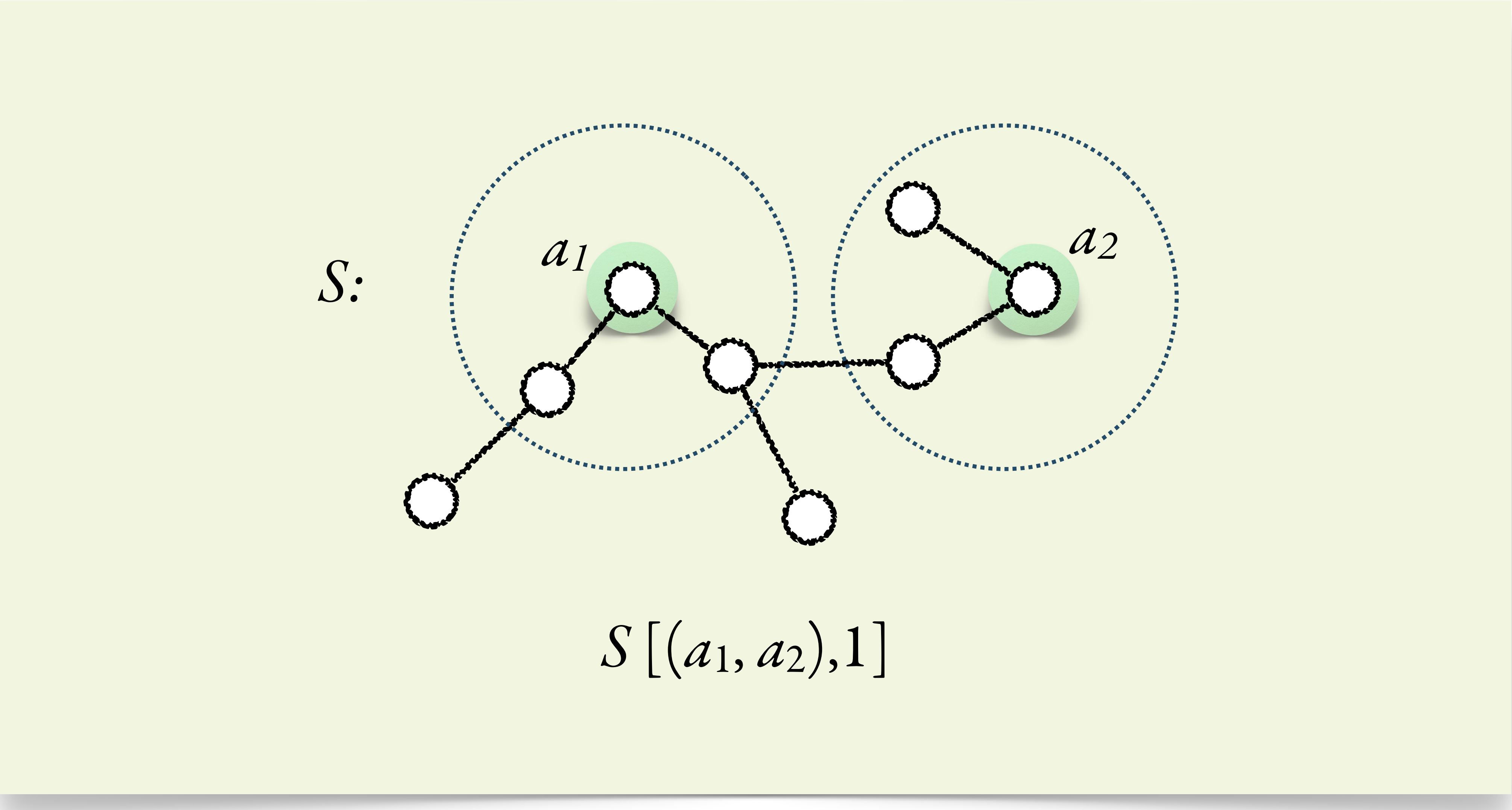
Eg: Is reachability expressible in FO?

What about equivalence on the same structure?

When are two points indistinguishable?

Gaifman locality

$S\,[(a_1, \ldots, a_n), r]$ = induced substructure of S of elements at distance $\leq r$ of some a_i in the Gaifman graph.



Gaifman locality

$S[(a_1, \ldots, a_n), r]$ = induced substructure of S

of elements at distance $\leq r$ of some a_i in the Gaifman graph.

Gaifman locality

For any $\phi \in$ FO of quantifier rank k and structure S,

$$S[(a_1, \ldots, a_n), r] \cong S[(b_1, \ldots, b_n), r] \quad \text{for } r = 3^{k+1}$$

implies

$$(a_1, \ldots, a_n) \in \phi(S) \quad \text{iff} \quad (b_1, \ldots, b_n) \in \phi(S)$$

Idea: If the neighbourhoods of two tuples are the same,
the formula cannot distinguish them.

Gaifman locality vs Hanf locality

Difference between Hanf- and Gaifman-locality:

Hanf-locality relates **two different structures,**

S_1 and S_2 have the same # of balls of radius 3^k, **up to threshold k**

$\Downarrow$

They verify the same **sentences** of qr $\leq k$

Gaifman-locality talks about definability in **one structure**

Inside S,

3^{k+1}-balls of $(a_1,\ldots,a_n)$ = 3^{k+1}-balls of $(b_1,\ldots,b_n)$

$\Downarrow$

$(a_1,\ldots,a_n)$ indistinguishable from $(b_1,\ldots,b_n)$ through **formulas** of qr $\leq k$

Gaifman locality

Schema to show non-expressibility results is, as usual:

A query $Q(x_1,\dots,x_n)$ is not FO-definable if:

for every $\boldsymbol{k}$ there is a structure $S_{\boldsymbol{k}}$ and $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$ such that

- $S_{\boldsymbol{k}}[(a_1, \dots, a_n), 3^{\boldsymbol{k}+1}] \cong S_{\boldsymbol{k}}[(b_1, \dots, b_n), 3^{\boldsymbol{k}+1}]$
- $(a_1, \dots, a_n) \in Q(S_{\boldsymbol{k}})$, $(b_1, \dots, b_n) \notin Q(S_{\boldsymbol{k}})$

Proof: If Q were expressible with a formula of quantifier rank k,

then $(a_1, \dots, a_n) \in Q(S_{\boldsymbol{k}})$ iff $(b_1, \dots, b_n) \in Q(S_{\boldsymbol{k}})$. Absurd!

Gaifman locality

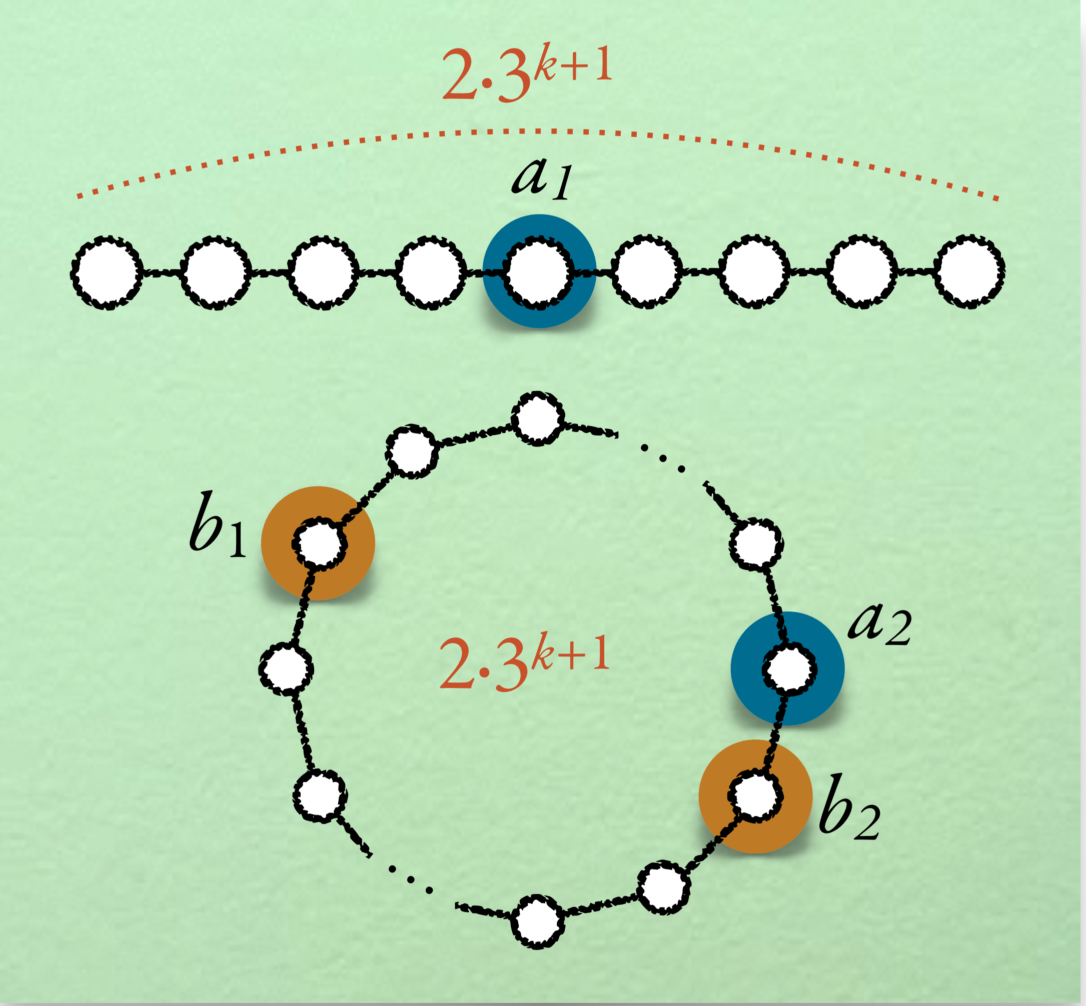
Reachability is not FO definable.

For every k, we build S_k :

And $S_k[(a_1, a_2), 3^{k+1}] \cong S_k[(b_1, b_2), 3^{k+1}]$

However,

- b_2 is reachable from b_1,
- a_2 is **not** reachable from a_1.

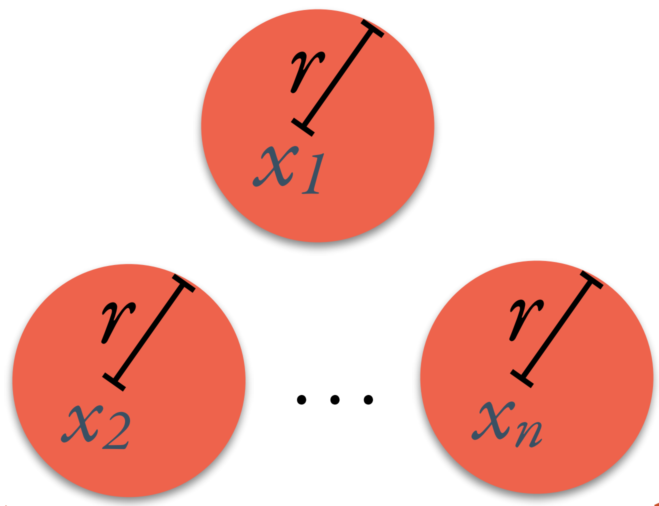


Your turn! Q(x) = "x is a vertex separator"

Gaifman Theorem

Basic local sentence:

$\exists\, x_1, \dots, x_n$



disjoint r-balls around $x_1, \dots, x_n$

$\wedge\ \psi_1(x_1) \wedge \cdots \wedge \psi_n(x_n)$

r-local formulas

Inside $\psi_i(x_i)$ we interpret
$\exists y \,.\, \phi$ as $\exists y \,.\, \mathrm{d}(x_i, y) \leq r \wedge \phi$

Gaifman Theorem: Every FO sentence is equivalent to
a boolean combination of **basic local sentences**.

Recap

EF games

FO sentences with quantifier rank n

=

winning strategies for Spoiler in the n-round EF game

0-1 Law

FO sentences are almost always true or almost always false

Hanf locality

FO sentences with quantifier rank n

=

counting 3^n sized balls up to n

Gaifman locality

Queries of quantifier rank n output tuples closed under 3^{n+1} balls.

Gaifman Theorem

An FO sentence can only say

"there are some points at distance ≥2r

whose r-balls are isomorphic to certain structures"

or a boolean combination of that.

Some more cool stuff…

Descriptive complexity

What properties can be checked efficiently? E.g. 3COL can be tested in NP

Metatheorem

"A property can be expressed in [insert some logic here]
iff
it can be checked in [some complexity class here]"

⇝ "A property is FO-definable iff it can be tested in AC^0"

⇝ "A property is ∃SO-definable iff it can be tested in NP" [Fagin 73]

⇝ Open problem: which logic captures PTIME?

Some more cool stuff…

Recursion

Can we enhance query languages with recursion ? E.g. express reachability properties

Datalog (semantics based on least fixpoint)

```
Ancestor(X,Y) :- Parent(X,Z), Ancestor(Z,Y)
Ancestor(X,X) :- .
?- Ancestor("Louis XIV",Y)
```

⇝ Incomparable with FO (has recursion, but is monotone)

⇝ Evaluation is in PTIME (for data complexity, but also for bounded arity)

Semi-structured data

Tree-structured or graph-structures dbs in place of relational dbs.

XML, XPath, Stream processing, …

```
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    <author>Matthew Gambardella</author>
    <year>2000</year>
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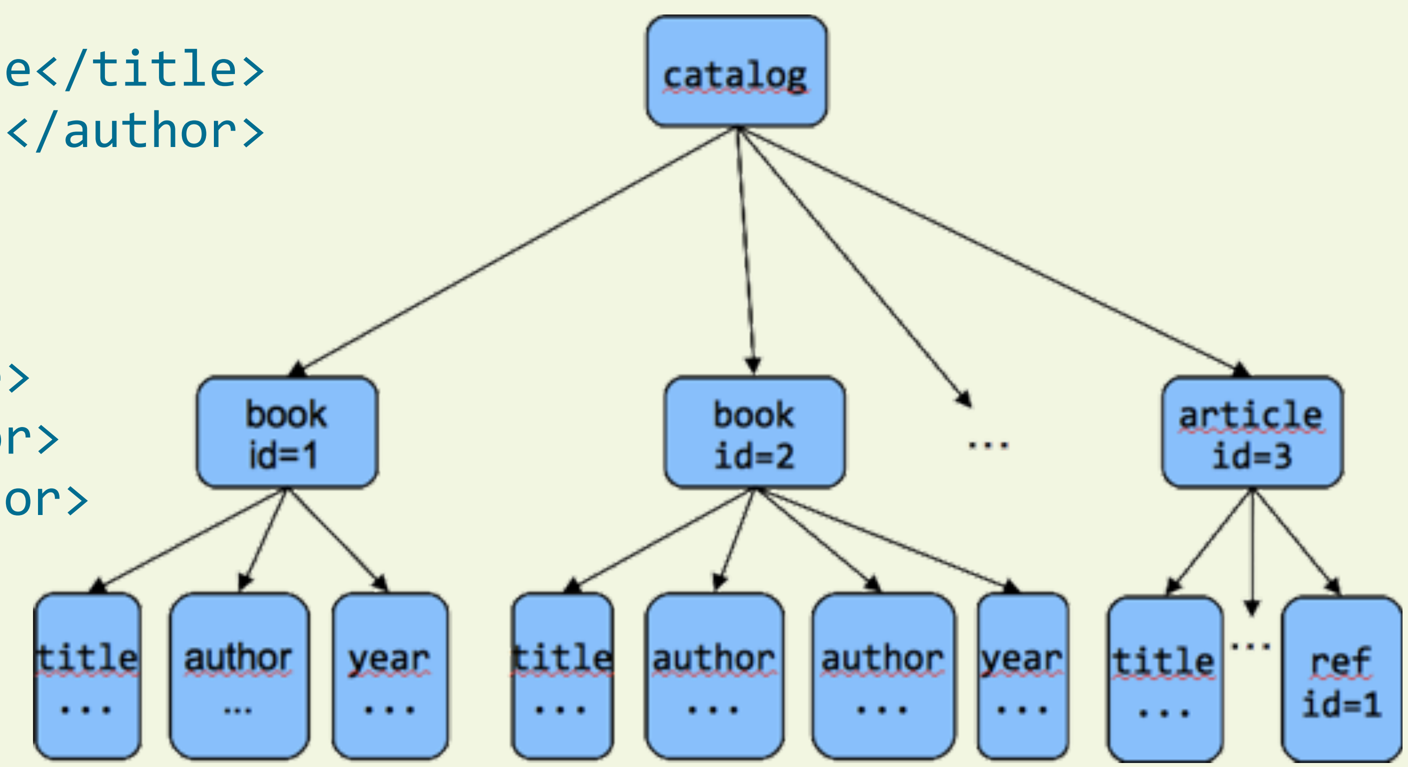


⇝ Evaluation of XPath is in linear time (data complexity) [Bojanczyk, Parys 08]

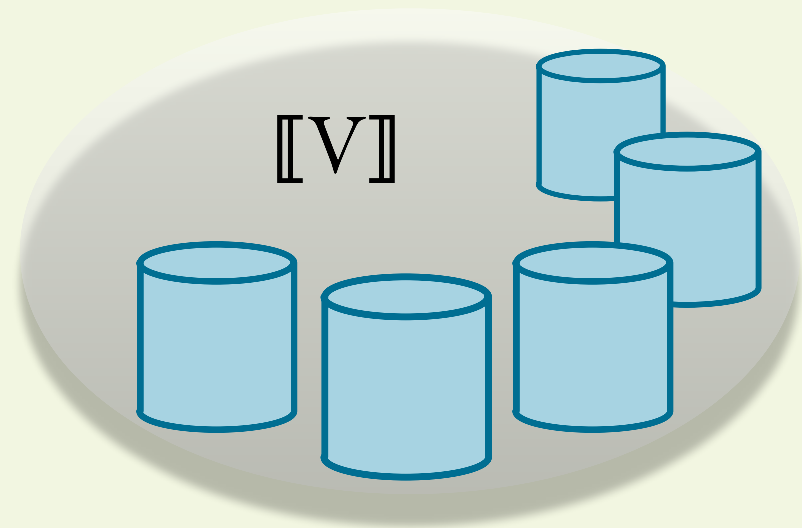
⇝ Satisfiability for $FO^2[\downarrow,\sim]$ is decidable [Bojanczyk, Muscholl, Schwentick, Segoufin 09]

Incomplete information

How to correctly reason when information is hidden/missing/noisy/… ?

Certain Query Answers (CQA)

V



$$\phi[\![V]\!] = \bigcap_{D \in [\![V]\!]} \phi(D)$$

⇝ CQA computable in PTIME w.r.t. view size. [Abiteboul, Kanellakis, Grahne 91]

Bibliography

- Abiteboul, Hull, Vianu, "Foundations of Databases", Addison-Wesley, 1995.
  (available at http://webdam.inria.fr/Alice/)
- Libkin, "Elements of Finite Model Theory", Springer, 2004.
- Immerman, "Descriptive Complexity", Springer, 1999.
- Otto, "Finite Model Theory", Springer, 2005
  (available at www.mathematik.tu-darmstadt.de/~otto/LEHRE/FMT0809.ps)
- Väänänen, "A Short course on Finite Model Theory", 1994.
  (available at www.math.helsinki.fi/logic/people/jouko.vaananen/shortcourse.pdf)