ESSLLI Tutorial: Nonmonotonic Logic

Approaches based on Maximal Consistent Subsets

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Aims of this session

1. Get familiar with approaches to nonmonotonic logic based on maximal consistent subsets

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1. Get familiar with approaches to nonmonotonic logic based on maximal consistent subsets
2. get familiar with or recall meta-theoretic principles that play an important role in nonmonotonic logic

Nonmonotonic Logics based on Maximal Consistent Subsets

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Some background

… as known from:

- Rescher / Manor consequence relations (Rescher and Manor (1970))
- Brewka's preferred subtheories (Brewka (1989))
- Benferhat, Dubious, Prade (Benferhat et al. (1997))
- Makinsons' Default Assumptions (Makinson (2003))
- Batens' Adaptive Logics and generalizations (Batens (2007); Straßer (2014); Van De Putte (2013))
- Constrained Input/Output logics (Makinson and Van Der Torre (2001))
- there are also close connections to argumentation-based methods (Amgoud and Besnard (2013))

The setup

(non-defeasible) facts

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- e.g., Σ_1 stems from the most reliable (though fallible) source, Σ_2 stems from the second most reliable (though fallible) source, etc.

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- e.g., Σ_1 stems from the most reliable (though fallible) source, Σ_2 stems from the second most reliable (though fallible) source, etc.
- in Rescher/Manor consequence relations: $\Sigma_0 = \emptyset$
- Makinsons's default assumptions: Σ_0 may be non-empty

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- cut: where $\Gamma' \subseteq \mathrm{Cn}(\Gamma)$ and $A \in \mathrm{Cn}(\Gamma \cup \Gamma')$, $A \in \mathrm{Cn}(\Gamma)$
- or, transitivity: where $\Gamma' \subseteq \mathrm{Cn}(\Gamma)$ and $A \in \mathrm{Cn}(\Gamma')$, $A \in \mathrm{Cn}(\Gamma)$.

Note

Given reflexivity and monotonicity, the following are equivalent:

- cut: where $\Gamma' \subseteq \mathrm{Cn}(\Gamma)$ and $A \in \mathrm{Cn}(\Gamma \cup \Gamma')$, $A \in \mathrm{Cn}(\Gamma)$
- transitivity: where $\Gamma' \subseteq \mathrm{Cn}(\Gamma)$ and $A \in \mathrm{Cn}(\Gamma')$, $A \in \mathrm{Cn}(\Gamma)$.

Proof.

Suppose $\Gamma' \subseteq \mathrm{Cn}(\Gamma)$.

- ($\Rightarrow$) Suppose $A \in \mathrm{Cn}(\Gamma')$.
- By Monotonicity, $A \in \mathrm{Cn}(\Gamma \cup \Gamma')$.
- By Cut, $A \in \mathrm{Cn}(\Gamma)$.
- ($\Leftarrow$) Suppose $A \in \mathrm{Cn}(\Gamma \cup \Gamma')$.
- By Reflexivity, $\Gamma \cup \Gamma' \subseteq \mathrm{Cn}(\Gamma)$.
- By Transitivity, $A \in \mathrm{Cn}(\Gamma)$.

(In)consistency

There are many notions of inconsistency of a set of formulas Γ, e.g.,

1. $\Gamma \vdash \bot$
2. $\Gamma \vdash p \wedge \neg p$ for a propositional atom p
3. $\Gamma \vdash A$ for all wff A
4. etc.

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In what follows we suppose option 1.

We call a set of formulas consistent if it is not inconsistent.

Maximal Consistent Subsets

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We write $\mathrm{MCS}(\mathbf{\Sigma})$ for the set of all maximal consistent subsets of $\mathbf{\Sigma}$.

We write $\mathrm{CS}(\mathbf{\Sigma})$ for the set of all Ξ for which item 1 holds, i.e., for the set of all consistent subsets of $\mathbf{\Sigma}$.

Examples

Let $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$ where

- $\Sigma_0 = \{s\}$
- $\Sigma_d = \{s \supset (p \land q), p \land \neg q, r\}$

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Existence of MCSs

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- Let $\Sigma_d = \{A_1, \ldots\}$. Let $\Xi' = \bigcup_{i \geq 0} \Xi_i$ where $\Xi_0 = \Xi$

$$\Xi_{i+1} = \begin{cases} \Xi_i \cup \{A_{i+1}\} & \text{if } \Xi_i \cup \{A_{i+1}\} \cup \Sigma_0 \nvdash \bot \\ \Xi_i & \text{else} \end{cases}$$

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- Assume $\Xi' \cup \Sigma_0 \vdash \bot$.
  - By compactness, there is a finite $\Xi'_f \subseteq \Xi'$ such that $\Xi'_f \cup \Sigma_0 \vdash \bot$.
  - Thus there is a $\Xi_i \supseteq \Xi'_f$ and by monotonicity, $\Xi_i \cup \Xi_0 \vdash \bot$,—a contradiction.
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- Hence, $\Xi' \cup \Sigma_0 \nvdash \bot$.
- Where $A_i \notin \Xi'$, $\Xi' \cup \{A_i\} \cup \Sigma_0 \vdash \bot$ since $\Xi_{i-1} \cup \{A_i\} \cup \Sigma_0 \vdash \bot$ and by monotonicity. □

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Defining Consequence Relations

General Idea

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Here we put at use maximal consistent subsets.

Three central consequence relations

Given $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, we define:

- Free consequences:

  $$\mathbf{\Sigma} \vdash_{\mathrm{free}} A \text{ iff } \Sigma_0 \cup \mathrm{Free}(\mathbf{\Sigma}) \vdash A \text{ where } \mathrm{Free}(\mathbf{\Sigma}) = \bigcap \mathrm{MCS}(\mathbf{\Sigma})$$

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- Existential consequences:

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Universal Consequences

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- *floating conclusion*: p

Question: Is there also some syntax-dependency for $\vdash_\forall$?

Take $\mathbf{\Sigma}' = \langle \emptyset, \{p \wedge q, \neg p\} \rangle$

- $\mathrm{MCS}(\mathbf{\Sigma}') = \big\{ \{p \wedge q\}, \{\neg p\} \big\}$
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While, where $\mathbf{\Sigma}'' = \langle \emptyset, \{p, q, \neg p\} \rangle$

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Existential Consequences

- $\mathbf{\Sigma} \vdash_{\exists} A$ iff $A \in \mathrm{Cn}(\Xi_1 \cup \{s\}) \cup \mathrm{Cn}(\Xi_2 \cup \{s\})$

Exercises

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4. The same holds for the universal consequences.
5. The free consequences are consistent.

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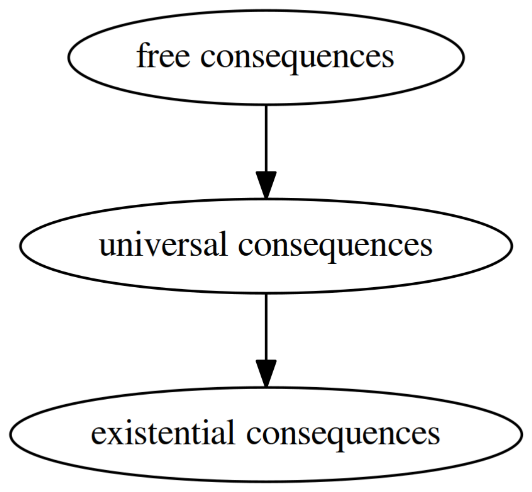
4. The same holds for the universal consequences.
5. The free consequences are consistent.
6. So are the universal consequences.

A hierarchy


free consequences
universal consequences
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Nonmonotonic Logics based on Maximal Consistent Subsets

What about stratified premise sets?

Making use of Lexicographic Orders

- Let $\langle \Sigma_0, \Sigma_d \rangle$ where $\Sigma_d = \langle \Sigma_1, \Sigma_2, \ldots \rangle$.
- Idea: Order MCSs relative to the strength of their constituting premises where strength/reliability/etc. is indirectly proportional to the index Σ_i.

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Lexicographic Ordering (e.g., Brewka (1989))

$\Xi \prec \Xi'$ iff there is an $i \geq 1$ such that

1. $\Xi \cap \Sigma_j = \Xi' \cap \Sigma_j$ for all $1 \leq j < i$, and
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Consequence relation

… relative to $\min_{\prec}(\mathrm{MCS}(\Sigma))$

- $\Sigma \vdash^{\prec}_{\mathrm{free}} A$ iff $\bigcap \min_{\prec}(\mathrm{MCS}(\Sigma)) \vdash A$
- $\Sigma \vdash^{\prec}_{\forall} A$ iff for all $\Xi \in \min_{\prec}(\mathrm{MCS}(\Sigma))$, $\Xi \vdash A$
- $\Sigma \vdash^{\prec}_{\exists} A$ iff for some $\Xi \in \min_{\prec}(\mathrm{MCS}(\Sigma))$, $\Xi \vdash A$

Time for an example

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Example

- $\Sigma_0 = \{s\}$
- $\Sigma_1 = \{s \supset p\}$
- $\Sigma_2 = \{\neg p, \neg q\}$
- $\Sigma_3 = \{q, r\}$

MCSs

- $\Xi_1 = \{s \supset p, \neg q, r\}$
- $\Xi_2 = \{s \supset p, q, r\}$
- $\Xi_3 = \{\neg p, \neg q, r\}$
- $\Xi_4 = \{\neg p, q, r\}$

order: $\Xi_1 \prec \Xi_2 \prec \Xi_3 \prec \Xi_4$

Caution: Other orders may not be smooth!

Suppose we associate strength in direct proporitionality to the index of Σ_i (e.g., premises in Σ_{i+1} are more reliably than premises in Σ_i ($i \geq 1$)).

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A problematic example

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Example

- Take $\Sigma = \langle \Sigma_0, \Sigma_1, \Sigma_2, \ldots \rangle$ where
- $\Sigma_0 = \{p_i \vee p_j \mid j > i \geq 1\}$
- $\Sigma_i = \{\neg p_i, s\}$ for each $i \geq 1$

MCSs

A problematic example

Ordering

$\Xi \sqsubset \Xi'$ iff there there is an $i \geq 1$ such that

1. $\Xi \cap \Sigma_j = \Xi' \cap \Sigma_j$ for all $j > i$, and
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MCSs

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MCSs

- Σ_i for each $i \geq 1$

Note

- $\min_{\sqsubset}(\mathrm{MCS}(\Sigma)) = \emptyset$
- however, we would at least expect to derive s

Some Meta-Theory

Some Meta-Theory

Monotonicity & Co

Monotonicity?

We can distinguish between (where $\mid\!\sim \in \{\vdash_{\forall}, \vdash_{\exists}, \vdash_{\mathrm{free}}\}$):

Monotonicity in the defeasible premises

If $\langle \Sigma_0, \Sigma_d \rangle \mid\!\sim A$ then $\langle \Sigma_0, \Sigma_d \cup \Gamma \rangle \mid\!\sim A$.

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If $\langle \Sigma_0, \Sigma_d \rangle \mathrel{|\!\!\sim} A$ then $\langle \Sigma_0, \Sigma_d \cup \Gamma \rangle \mathrel{|\!\!\sim} A$.

Monotonicity in the factual premises

If $\langle \Sigma_0, \Sigma_d \rangle \mathrel{|\!\!\sim} A$ then $\langle \Sigma_0 \cup \Gamma, \Sigma_d \rangle \mathrel{|\!\!\sim} A$.

Can you think of a counter-example to monotonicity for our consequence relations?

Exercise

Prove that monotonicity holds for existential consequences in the defeasible premises.

Some Meta-Theory

Weakening Monotonicity

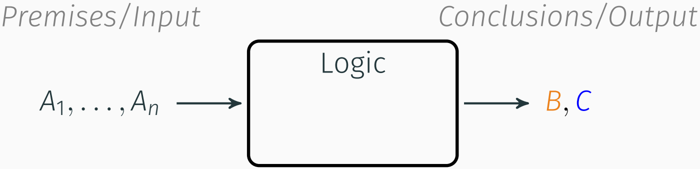
Recall that there are weakened forms of monotonicity, most prominently the following two principles …

Cautious Monotonicity

If $A_1, \ldots, A_n \mathrel{|\!\sim} B$ and $A_1, \ldots, A_n \mathrel{|\!\sim} C$, then $A_1, \ldots, A_n, B \mathrel{|\!\sim} C$.

Cautious Monotonicity

If $A_1, \ldots, A_n \mathrel{|\!\sim} B$ and $A_1, \ldots, A_n \mathrel{|\!\sim} C$, then $A_1, \ldots, A_n, B \mathrel{|\!\sim} C$.



"We do not loose conclusions if we plug in conclusions as additional input."

Cautious Monotonicity

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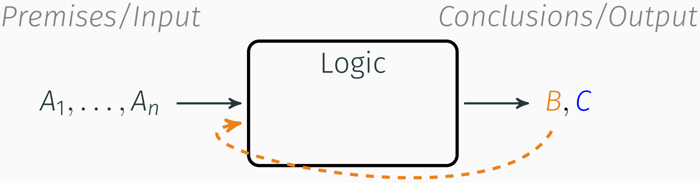


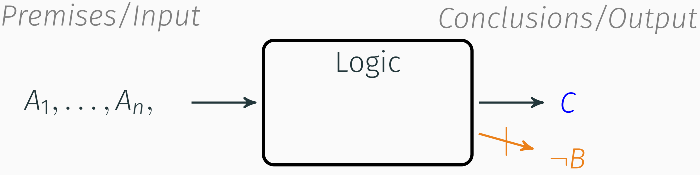
"We do not loose conclusions if we plug in conclusions as additional input."

Rational Monotonicity

If it is not the case that $A_1, \ldots, A_n \mid\!\sim \neg B$, and moreover $A_1, \ldots, A_n \mid\!\sim C$, then $A_1, \ldots, A_n, B \mid\!\sim C$.

Rational Monotonicity

If it is *not* the case that $A_1, \ldots, A_n \mid\!\sim \neg B$, and moreover $A_1, \ldots, A_n \mid\!\sim C$, then $A_1, \ldots, A_n, B \mid\!\sim C$.



"Our conclusions are robust under that addition of consistent information to the input."

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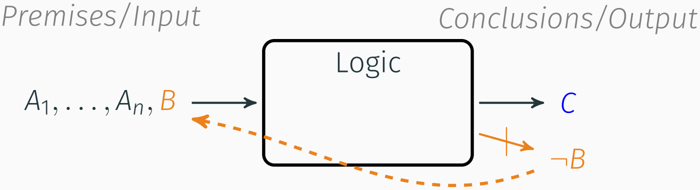

"Our conclusions are robust under that addition of consistent information to the input."

Some Meta-Theory

Cumulativity

We will treat

Cautious Monotonicity

If $\Sigma \mid\!\sim A$ and $\Sigma \mid\!\sim B$ then $\Sigma \cup \{A\} \mid\!\sim B$.

and

Cut

If $\Sigma \mid\!\sim A$ and $\Sigma \cup \{A\} \mid\!\sim B$ then $\Sigma \mid\!\sim B$.

at one single blow ...

Cumulativity

If $\Sigma \mid\!\sim A$: $\Sigma \mid\!\sim B$ iff $\Sigma \cup \{A\} \mid\!\sim B$

Cumulativity

$$\text{If } \Sigma \mid\!\sim A : \ \Sigma \mid\!\sim B \ \text{ iff } \ \Sigma \cup \{A\} \mid\!\sim B$$

- Cautious Monotonicity

Cumulativity

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- Cut

Cumulativity

$$\text{If } \Sigma \mathrel{|\!\sim} A : \Sigma \mathrel{|\!\sim} B \text{ iff } \Sigma \cup \{A\} \mathrel{|\!\sim} B$$

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- Cut

Cumulativity in defeasible premises:

$$\text{If } \langle \Sigma_0, \Sigma_d \rangle \mathrel{|\!\sim} A : \langle \Sigma_0, \Sigma_d \rangle \mathrel{|\!\sim} B \text{ iff } \langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \mathrel{|\!\sim} B$$

Cumulativity in factual premises:

$$\text{If } \langle \Sigma_0, \Sigma_d \rangle \mathrel{|\!\sim} A : \langle \Sigma_0, \Sigma_d \rangle \mathrel{|\!\sim} B \text{ iff } \langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \mathrel{|\!\sim} B$$

Some Meta-Theory

Cumulativity in the factual premises

For this, proving one central lemma will be sufficient ...

A central lemma

Note that

Cumulativity in the factual premises ($\mid\!\sim\, \in \{\vdash_\forall, \vdash_{\text{free}}\}$)

$$\text{If } \langle \Sigma_0, \Sigma_d \rangle \mid\!\sim A : \langle \Sigma_0, \Sigma_d \rangle \mid\!\sim B \text{ iff } \langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \mid\!\sim B$$

follows immediately with the following lemma:

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follows immediately with the following lemma:

Lemma 1

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$, *then* $\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) = \mathrm{MCS}(\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle)$.

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(Simple) Exercise

Try to prove that Cumulativity for the different consequence relations follows from Lemma 1.

A central lemma (cont.)

Lemma 1

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$, then $\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) = \mathrm{MCS}(\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle)$.

Proof of ”$\subseteq$”.

A central lemma (cont.)

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If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$, then $\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) = \mathrm{MCS}(\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle)$.

Proof of "$\subseteq$".

Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$ and let $(\dagger)$ $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$.

A central lemma (cont.)

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If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$, then $\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) = \mathrm{MCS}(\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle)$.

Proof of "$\subseteq$".

Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$ and let (†) $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$.

- Thus, $\Xi \cup \Sigma_0 \vdash A$ and hence, by (†) and cut,

$$\Sigma_0 \cup \Xi \cup \{A\} \nvdash \bot \tag{1}$$

A central lemma (cont.)

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If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$, then $\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) = \mathrm{MCS}(\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle)$.

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- Assume for a contradiction that there is a $\Xi' \in \mathrm{MCS}(\Sigma_0 \cup \{A\}, \Sigma_d)$ such that $\Xi \subset \Xi'$.

A central lemma (cont.)

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- Assume for a contradiction that there is a $\Xi' \in \mathrm{MCS}(\Sigma_0 \cup \{A\}, \Sigma_d)$ such that $\Xi \subset \Xi'$.
- Hence, $\Sigma_0 \cup \{A\} \cup \Xi' \nvdash \bot$ and hence $\Sigma_0 \cup \Xi' \nvdash \bot$ by monotonicity.

A central lemma (cont.)

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If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$, then $\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) = \mathrm{MCS}(\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle)$.

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- Hence, $\Sigma_0 \cup \{A\} \cup \Xi' \nvdash \bot$ and hence $\Sigma_0 \cup \Xi' \nvdash \bot$ by monotonicity.
- This contradicts that $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$.

A central lemma (cont.)

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If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$, then $\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) = \mathrm{MCS}(\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle)$.

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- Hence, $\Sigma_0 \cup \{A\} \cup \Xi' \nvdash \bot$ and hence $\Sigma_0 \cup \Xi' \nvdash \bot$ by monotonicity.
- This contradicts that $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$.
- Thus, by (1), $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$.

Exercise

Exercise

Proof the other direction of Lemma 1.

Some Meta-Theory

Cumulativity in the defeasible premises

For this, proving one central lemma will be sufficient …

(sounds familiar?)

A lemma central to Cumulativity

Lemma 2

Where $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\mathbf{\Sigma} \vdash_\forall A$,
$\pi : \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \rightarrow \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle), \Lambda \mapsto \Lambda \cup \{A\}$ *is a bijection.*

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Proof.

We proceed in steps …

First we exclude the trivial case …

Excluding the trivial case

Lemma 2

Where $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\mathbf{\Sigma} \vdash_\forall A$,
$\pi : \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \to \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle), \Lambda \mapsto \Lambda \cup \{A\}$ is a bijection.

Proof (cont.)

If $A \in \Sigma_d$ the lemma is trivial. Suppose thus that $A \notin \Sigma_d$.

Next, we should check whether π is well-defined.

Is π well-defined?

Proof (cont.)

- Let $\Xi \in \mathrm{MCS}(\boldsymbol{\Sigma})$.

Is π well-defined?

Proof (cont.)

- Let $\Xi \in \mathrm{MCS}(\boldsymbol{\Sigma})$.
- We show that $\Xi \cup \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.

Is π well-defined?

Proof (cont.)

- Let $\Xi \in \mathrm{MCS}(\mathbf{\Sigma})$.
- We show that $\Xi \cup \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.
- Note that $\Xi \cup \Sigma_0 \vdash A$. Thus, by cut and since $\Xi \in \mathrm{MCS}(\mathbf{\Sigma})$,

$$\Sigma_0 \cup \Xi \cup \{A\} \nvdash \bot \tag{2}$$

Is π well-defined?

Proof (cont.)

- Let $\Xi \in \mathrm{MCS}(\boldsymbol{\Sigma})$.
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$$\Sigma_0 \cup \Xi \cup \{A\} \nvdash \bot \tag{2}$$

- Assume for a contradiction that there is a $\Xi' \supset \Xi \cup \{A\}$ such that $\Xi' \in \mathrm{MCS}(\Sigma_0, \Sigma_d \cup \{A\})$.

Is π well-defined?

Proof (cont.)

- Let $\Xi \in \mathrm{MCS}(\mathbf{\Sigma})$.
- We show that $\Xi \cup \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.
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- Assume for a contradiction that there is a $\Xi' \supset \Xi \cup \{A\}$ such that $\Xi' \in \mathrm{MCS}(\Sigma_0, \Sigma_d \cup \{A\})$.
- Thus, $\Sigma_d \supseteq \Xi' \setminus \{A\} \supset \Xi$.

Is π well-defined?

Proof (cont.)

- Let $\Xi \in \mathrm{MCS}(\mathbf{\Sigma})$.
- We show that $\Xi \cup \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.
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- Assume for a contradiction that there is a $\Xi' \supset \Xi \cup \{A\}$ such that $\Xi' \in \mathrm{MCS}(\Sigma_0, \Sigma_d \cup \{A\})$.
- Thus, $\Sigma_d \supseteq \Xi' \setminus \{A\} \supset \Xi$.
- Since $\Sigma_0 \cup (\Xi' \setminus \{A\}) \nvdash \bot$ (by monotonicity), this is a contradiction to Ξ being in $\mathrm{MCS}(\mathbf{\Sigma})$.

Is π well-defined?

Proof (cont.)

- Let $\Xi \in \mathrm{MCS}(\boldsymbol{\Sigma})$.
- We show that $\Xi \cup \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.
- Note that $\Xi \cup \Sigma_0 \vdash A$. Thus, by cut and since $\Xi \in \mathrm{MCS}(\boldsymbol{\Sigma})$,

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- Assume for a contradiction that there is a $\Xi' \supset \Xi \cup \{A\}$ such that $\Xi' \in \mathrm{MCS}(\Sigma_0, \Sigma_d \cup \{A\})$.
- Thus, $\Sigma_d \supseteq \Xi' \setminus \{A\} \supset \Xi$.
- Since $\Sigma_0 \cup (\Xi' \setminus \{A\}) \nvdash \bot$ (by monotonicity), this is a contradiction to Ξ being in $\mathrm{MCS}(\boldsymbol{\Sigma})$.
- Thus, the assumption is wrong and in view of (2), $\Xi \cup \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.

Next, we show that π is surjective.
(Recall, this means that every entity in the co-domain of π is in the image of π.)

π is surjective

Proof (cont.)

We now show that π is surjective. Let for this (‡)
$\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.

Proof (cont.)

We now show that π is surjective. Let for this (‡) $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.

- By monotonicity and since $\Sigma_0 \cup \Xi \nvdash \bot$, also

$$\Sigma_0 \cup (\Xi \setminus \{A\}) \nvdash \bot \tag{3}$$

Proof (cont.)

We now show that π is surjective. Let for this (‡) $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.

- By monotonicity and since $\Sigma_0 \cup \Xi \nvdash \bot$, also

$$\Sigma_0 \cup (\Xi \setminus \{A\}) \nvdash \bot \tag{3}$$

- Assume for a contradiction that there is a $\Xi' \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$ such that $\Xi \setminus \{A\} \subset \Xi' \subseteq \Sigma_d$.

π is surjective

Proof (cont.)

We now show that π is surjective. Let for this (‡) $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.

- By monotonicity and since $\Sigma_0 \cup \Xi \nvdash \bot$, also

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- Assume for a contradiction that there is a $\Xi' \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$ such that $\Xi \setminus \{A\} \subset \Xi' \subseteq \Sigma_d$.
- By the supposition, $\Sigma_0 \cup \Xi' \vdash A$ and hence $\Sigma_0 \cup \Xi' \cup \{A\} \nvdash \bot$.

π is surjective

Proof (cont.)

We now show that π is surjective. Let for this (‡) $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.

- By monotonicity and since $\Sigma_0 \cup \Xi \nvdash \bot$, also

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- Assume for a contradiction that there is a $\Xi' \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$ such that $\Xi \setminus \{A\} \subset \Xi' \subseteq \Sigma_d$.
- By the supposition, $\Sigma_0 \cup \Xi' \vdash A$ and hence $\Sigma_0 \cup \Xi' \cup \{A\} \nvdash \bot$.
- This is a contradiction to (‡) since $\Xi \subset \Xi' \cup \{A\} \subseteq \Sigma_d \cup \{A\}$.

π is surjective

Proof (cont.)

We now show that π is surjective. Let for this (‡) $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.

- By monotonicity and since $\Sigma_0 \cup \Xi \nvdash \bot$, also

$$\Sigma_0 \cup (\Xi \setminus \{A\}) \nvdash \bot \tag{3}$$

- Assume for a contradiction that there is a $\Xi' \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$ such that $\Xi \setminus \{A\} \subset \Xi' \subseteq \Sigma_d$.
- By the supposition, $\Sigma_0 \cup \Xi' \vdash A$ and hence $\Sigma_0 \cup \Xi' \cup \{A\} \nvdash \bot$.
- This is a contradiction to (‡) since $\Xi \subset \Xi' \cup \{A\} \subseteq \Sigma_d \cup \{A\}$.
- Since our assumption is wrong and by (3), $\Xi \setminus \{A\} \in \mathrm{MCS}(\mathbf{\Sigma})$.

What is left, is to show that π is injective.
Fortunately, this is easy ...

π is injective

Proof (cont.)

That π is injective is trivial in view of the supposition that $A \notin \Sigma_d$. □

So, how does this help in proving Cumulativity?
Let’s see and start with $\vdash_\forall$ …

Cumulativity

Theorem 3

Where $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$*:* $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ *iff* $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$.

We again proceed in steps:

1. we exclude a trivial case
2. we proof the ($\Rightarrow$)-direction
3. we proof the ($\Leftarrow$)-direction

This will all be pretty straight-forward, given our lemma.

Cumulativity (cont.): excluded a trivial case

Proof.

In case $A \in \Sigma_d$ the proposition is trivial. Suppose thus that $A \notin \Sigma_d$.

Cumulativity (cont.): the ($\Rightarrow$)-direction

Proof.

Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$.

- Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$.

Cumulativity (cont.): the ($\Rightarrow$)-direction

Proof.

Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$.

- Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$.
- To show: $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$.

Cumulativity (cont.): the ($\Rightarrow$)-direction

Proof.

Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$.

- Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$.
- To show: $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$.
- Let $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.

Cumulativity (cont.): the ($\Rightarrow$)-direction

Proof.

Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$.

- Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$.
- To show: $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$.
- Let $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.
- Hence, by our Lemma 2, $\Xi \setminus \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$ and thus $\Sigma_0 \cup (\Xi \setminus \{A\}) \vdash B$.

Cumulativity (cont.): the ($\Rightarrow$)-direction

Proof.

Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$.

- Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$.
- To show: $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$.
- Let $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.
- Hence, by our Lemma 2, $\Xi \setminus \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$ and thus $\Sigma_0 \cup (\Xi \setminus \{A\}) \vdash B$.
- By Monotonicity, $\Sigma_0 \cup \Xi \vdash B$.

Cumulativity (cont.): the ($\Leftarrow$)-direction

Proof.

- Suppose $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$.
- Let $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$.
- Hence, by our Lemma 2, $\Xi \cup \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$ and $\Sigma_0 \cup \Xi \vdash A$.
- Also, $\Sigma_0 \cup \Xi \cup \{A\} \vdash B$.
- By cut, $\Sigma_0 \cup \Xi \vdash B$.
- Thus, $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$.

$\square$

Exercise: Cumulativity for the Free Consequences

Note that with Lemma 2 the following follows immediately:

Lemma 4

Where $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$,

$\bigcap \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle) = \bigcap \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \cup \{A\}$

Exercise

Try to prove the following two theorems:

Theorem 5

Where $\langle \Sigma_0, \Sigma_d \rangle \vdash_{\mathrm{free}} A$*:* $\langle \Sigma_0, \Sigma_d \rangle \vdash_{\mathrm{free}} B$ *iff* $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_{\mathrm{free}} B$.

Theorem 6

Where $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$*:* $\langle \Sigma_0, \Sigma_d \rangle \vdash_{\mathrm{free}} B$ *iff* $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_{\mathrm{free}} B$.

Some Meta-Theory

A stronger form of Cautious Monotonicity

Interestingly, we can strengthen our previous result by adjusting our lemma a bit …

A strengthened Lemma

Recall our lemma:

Lemma 2

Where $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\mathbf{\Sigma} \vdash_\forall A$,
$\pi : \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \to \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle), \Lambda \mapsto \Lambda \cup \{A\}$ is a bijection.

A strengthened Lemma

Recall our lemma:

we're talking about this bit

Lemma 2

Where $\boldsymbol{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\boldsymbol{\Sigma} \vdash_\forall A$,
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Do you have an idea how to weaken the antecedent of this lemma?

A strengthened Lemma

Recall our lemma:

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Where $\boldsymbol{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\boldsymbol{\Sigma} \vdash_\forall A$,
$\pi : \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \to \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle), \Lambda \mapsto \Lambda \cup \{A\}$ is a bijection.

Do you have an idea how to weaken the antecedent of this lemma?

Lemma 7

Where $\Sigma \nvdash_\exists \neg A$,
$\pi : \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \to \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle), \Lambda \mapsto \Lambda \cup \{A\}$ is a bijection.

A strengthened Lemma

Recall our lemma:

we're talking about this bit

Lemma 2

Where $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\mathbf{\Sigma} \vdash_\forall A$,
$\pi : \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \to \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle), \Lambda \mapsto \Lambda \cup \{A\}$ is a bijection.

Do you have an idea how to weaken the antecedent of this lemma?

here we suppose to have a classical negation

Where $\Sigma \nvdash_\exists \neg A$,
$\pi : \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \to \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle), \Lambda \mapsto \Lambda \cup \{A\}$ *is a bijection.*

Exercise

Adjust the proof of Lemma 2 to prove Lemma 7.

A stronger form of Cautious Monotonicity

Our lemma immediately gives rise to the following theorem.

Theorem 8

Where $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\exists \neg A$*:* $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ *implies* $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$*.*

A stronger form of Cautious Monotonicity

Our lemma immediately gives rise to the following theorem.

Theorem 8

Where $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\exists \neg A$: $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ implies $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$.

Proof.

Suppose ~~$\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A$~~ $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\exists \neg A$.

- Suppose $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$.
- Let $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$.
- Hence, by our Lemma 7, $\Xi \setminus \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$ and thus $\Sigma_0 \cup (\Xi \setminus \{A\}) \vdash B$.
- By Monotonicity, $\Sigma_0 \cup \Xi \vdash B$.

□

Wait a moment, doesn't $\mathbf{\Sigma} \nvdash_{\exists} \neg A$ imply $\mathbf{\Sigma} \vdash_{\forall} A$ and are we not back to where we started?

Example

Here's an example to show that $\mathbf{\Sigma} \nvdash_{\exists} \neg A$ doesn't imply $\mathbf{\Sigma} \vdash_{\forall} A$.

Example

- Take $\mathbf{\Sigma} = \langle \emptyset, \emptyset \rangle$.
- We have one MCS: $\emptyset$
- Clearly,
  - $\mathbf{\Sigma} \nvdash_{\exists} \neg p$
  - $\mathbf{\Sigma} \nvdash_{\forall} p$

Example

Here's an example to show that $\mathbf{\Sigma} \nvdash_{\exists} \neg A$ doesn't imply $\mathbf{\Sigma} \vdash_{\forall} A$.

Example

- Take $\mathbf{\Sigma} = \langle \emptyset, \emptyset \rangle$.
- We have one MCS: $\emptyset$
- Clearly,
  - $\mathbf{\Sigma} \nvdash_{\exists} \neg p$
  - $\mathbf{\Sigma} \nvdash_{\forall} p$

Simple exercise

Show that $\mathbf{\Sigma} \vdash_{\forall} A$ implies $\mathbf{\Sigma} \nvdash_{\exists} \neg A$.

Some Meta-Theory

What about Rational Monotonicity?

Our strengthened Cautious Monotonicity is quite similar to Rational Monotonicity …

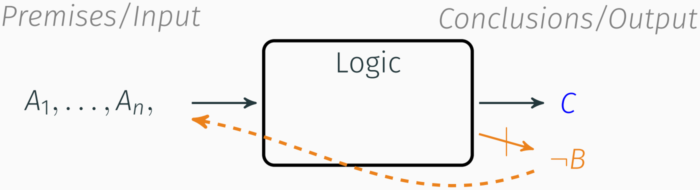
Theorem 8 and Rational Monotonicity

Theorem 8

Where $\langle \Sigma_0, \Sigma_d \rangle \nvdash_{\exists} \neg B$: $\langle \Sigma_0, \Sigma_d \rangle \vdash_{\forall} C$ implies $\langle \Sigma_0, \Sigma_d \cup \{B\} \rangle \vdash_{\forall} C$.

Recall our general form of Rational Monotonicity:

If it is not the case that $A_1, \ldots, A_n \mid\sim \neg B$, and moreover $A_1, \ldots, A_n \mid\sim C$, then $A_1, \ldots, A_n, B \mid\sim C$.



Theorem 8 and Rational Monotonicity

Theorem 8

Where $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\exists \neg B$: $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall C$ implies $\langle \Sigma_0, \Sigma_d \cup \{B\} \rangle \vdash_\forall C$.

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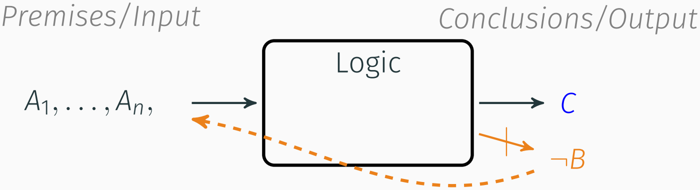

Theorem 8

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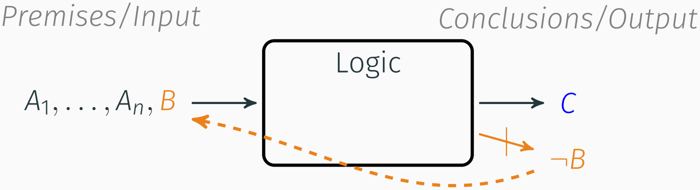

Should we be optimistic about Rational Monotonicity in view of this?

Two forms of Rational Monotonicity (relative to $\vdash_\forall$)

Rational Monotonicity in the defeasible premises

Where $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg B$: $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall C$ implies $\langle \Sigma_0, \Sigma_d \cup \{B\} \rangle \vdash_\forall C$.

Rational Monotonicity in the factual premises

Where $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg B$: $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall C$ implies $\langle \Sigma_0 \cup \{B\}, \Sigma_d \rangle \vdash_\forall C$.

Negative result for defeasible premises

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ and $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg A$ then $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$?

Negative result for defeasible premises

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ and $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg A$ then $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$?

- Nope! Take:
- $\Sigma_0 = \emptyset$
- $\Sigma_d = \{r, p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q, \neg p \wedge q\}$
- $A = p$ and $B = q$

Negative result for defeasible premises

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ and $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg A$ then $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$?

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- $A = p$ and $B = q$
- $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall q$
- $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg p$

Negative result for defeasible premises

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ and $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg A$ then $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$?

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- $\Sigma_0 = \emptyset$
- $\Sigma_d = \{r, p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q, \neg p \wedge q\}$
- $A = p$ and $B = q$
- $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall q$
- $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg p$
- but $\langle \Sigma_0, \Sigma_d \cup \{p\} \rangle \nvdash_\forall q$.
- Do you see why?

Negative result for defeasible premises

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ and $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg A$ then $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$?

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- but $\langle \Sigma_0, \Sigma_d \cup \{p\} \rangle \nvdash_\forall q$.
- Do you see why?

$\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) =$

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$\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) =$

1. $\{(p \wedge r) \supset \neg q, r, \neg p \wedge q\}$
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Negative result for defeasible premises

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ and $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg A$ then $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$?

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1. $\{(p \wedge r) \supset \neg q, r, \neg p \wedge q\}$
2. $\{(p \wedge r) \supset \neg q, p \wedge q \wedge \neg r, p\}$
3. $\{r, p, (p \wedge r) \supset \neg q\}$

Also Rational Monotonicity for factual premises doesn’t work out. We can use the same example to demonstrate this.

Negative result for factual premises

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ and $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg A$ then $\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \vdash_\forall B$?

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- $A = p$ and $B = q$
- $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall q$
- $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg p$

Negative result for factual premises

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ and $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg A$ then $\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \vdash_\forall B$?

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- $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall q$
- $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg p$
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- Do you see why?

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Negative result for factual premises

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- $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall q$
- $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg p$
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- Do you see why?

$\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) =$

1. $\{(p \wedge r) \supset \neg q, r, \neg p \wedge q\}$
2. $\{(p \wedge r) \supset \neg q, p \wedge q \wedge \neg r\}$

$\mathrm{MCS}(\langle \Sigma_0 \cup \{p\}, \Sigma_d \rangle) =$

Negative result for factual premises

If $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall B$ and $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg A$ then $\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \vdash_\forall B$?

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- $A = p$ and $B = q$
- $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall q$
- $\langle \Sigma_0, \Sigma_d \rangle \nvdash_\forall \neg p$
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$\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) =$

1. $\{(p \wedge r) \supset \neg q, r, \neg p \wedge q\}$
2. $\{(p \wedge r) \supset \neg q, p \wedge q \wedge \neg r\}$

$\mathrm{MCS}(\langle \Sigma_0 \cup \{p\}, \Sigma_d \rangle) =$

1. $\{r, (p \wedge r) \supset \neg q\}$
2. $\{p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q\}$

Exercise

Find similar counter-examples for Rational Monotonicity and $\vdash_{\text{free}}$.

Some Meta-Theory

The Resolution Theorem

Two versions:

Where $\mid\!\sim \in \{\vdash_\forall, \vdash_{\mathrm{free}}\}$ and we have an implication $\supset$ in our formal language,

The Resolution Theorem in the factual premises

$\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \mid\!\sim B$ if $\langle \Sigma_0, \Sigma_d \rangle \mid\!\sim A \supset B$.

The Resolution Theorem in the defeasible premises

$\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \mid\!\sim B$ if $\langle \Sigma_0, \Sigma_d \rangle \mid\!\sim A \supset B$.

What do you expect?

Given the centrality of this result in Classical Logic, do we get it in the nonmonotonic case (maybe at least where the base logic is Classical Logic)?

A negative result for the case of defeasible premises

Where $\mid\!\sim \in \{\vdash_\forall, \vdash_{\text{free}}\}$ and we have an implication $\supset$ in our formal language, the following does not hold (in general):

The Resolution Theorem in the defeasible premises

$\langle \Sigma_0, \Sigma_d \cup \{A\}\rangle \mid\!\sim B$ if $\langle \Sigma_0, \Sigma_d\rangle \mid\!\sim A \supset B$.

Counter-Example

- Consider $\mathbf{\Sigma} = \langle \emptyset, \{\neg p\}\rangle$, $A = p$ and $B = q$.
- Then $\mathbf{\Sigma} \mid\!\sim p \supset q$ while $\langle \emptyset, \{\neg p, p\}\rangle \not\mid\!\sim q$.

A negative result for the case of factual premises

Where $\mid\!\sim \in \{\vdash_{\forall}, \vdash_{\mathrm{free}}\}$ and we have an implication $\supset$ in our formal language, the following does not hold:

The Resolution Theorem in the factual premises

$\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \mid\!\sim B$ if $\langle \Sigma_0, \Sigma_d \rangle \mid\!\sim A \supset B$.

Counter-Example

- Take $\mathbf{\Sigma} = \langle \emptyset, \{\neg p\} \rangle$ and $\mathbf{\Sigma}^+ = \langle \{p\}, \{\neg p\} \rangle$.
- Then $\mathbf{\Sigma} \mid\!\sim p \supset q$ while $\mathbf{\Sigma}^+ \not\mid\!\sim q$.

Some Meta-Theory

The Deduction Theorem

Two versions:

Where $\mathrel{|\!\sim} \in \{\vdash_\forall, \vdash_{\mathrm{free}}\}$ and we have an implication $\supset$ in our formal language,

The deduction theorem in the factual premises

$\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \mathrel{|\!\sim} B$ implies $\langle \Sigma_0, \Sigma_d \rangle \mathrel{|\!\sim} A \supset B$.

The deduction theorem in the defeasible premises

$\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \mathrel{|\!\sim} B$ implies $\langle \Sigma_0, \Sigma_d \rangle \mathrel{|\!\sim} A \supset B$.

What do you think? We all know this is a central property in classical logic. It should hold here as well, at least if the base logic is classical, right?

The free case: a negative result for factual premises

We now give a counter-example to the following principle:

The deduction theorem in the factual premises

$\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \vdash_{\text{free}} B$ implies $\langle \Sigma_0, \Sigma_d \rangle \vdash_{\text{free}} A \supset B$.

The free case: a negative result for factual premises

We now give a counter-example to the following principle:

The deduction theorem in the factual premises

$\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \vdash_{\text{free}} B$ implies $\langle \Sigma_0, \Sigma_d \rangle \vdash_{\text{free}} A \supset B$.

Let

- $\Sigma_0 = \{p \vee q, r \vee q\}$
- $\Sigma_d = \{\neg p, \neg q\}$

We have

- $\langle \Sigma_0, \Sigma_d \rangle \nvdash_{\text{free}} p \supset r$ while
- $\langle \Sigma_0 \cup \{p\}, \Sigma_d \rangle \vdash_{\text{free}} r$

The free case: a negative result for factual premises

We now give a counter-example to the following principle:

The deduction theorem in the factual premises

$\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \vdash_{\text{free}} B$ implies $\langle \Sigma_0, \Sigma_d \rangle \vdash_{\text{free}} A \supset B$.

Let

- $\Sigma_0 = \{p \vee q, r \vee q\}$
- $\Sigma_d = \{\neg p, \neg q\}$

We have

- $\langle \Sigma_0, \Sigma_d \rangle \nvdash_{\text{free}} p \supset r$ while
- $\langle \Sigma_0 \cup \{p\}, \Sigma_d \rangle \vdash_{\text{free}} r$

To see this notice that

- $\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) = \{\{\neg p\}, \{\neg q\}\}$
- $\mathrm{MCS}(\langle \Sigma_0 \cup \{p\}, \Sigma_d \rangle) = \{\{\neg q\}\}$

Hm, so what’s your guess for the universal consequence: do we get the Deduction Theorem there?

A useful and simple lemma

Lemma 9

If $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$ *and* $\Sigma_0 \cup \Xi \cup \{A\} \nvdash \bot$ *then* $\Xi \in \mathrm{MCS}(\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle)$.

Exercise

Try to prove this.

Deduction Theorem: the universal case in the factual premises

We now suppose the Deduction Theorem holds for the Base Logic **L** and prove:

The deduction theorem in the factual premises

$\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \vdash_\forall B$ implies $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A \supset B$.

Proof.

- Suppose $\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle \vdash_\forall B$.
- Let $\Xi \in \mathrm{MCS}(\Sigma_0, \Sigma_d)$.
- If $\Sigma_0 \cup \Xi \cup \{A\} \vdash \bot$ also $\Sigma_0 \cup \Xi \cup \{A\} \vdash B$ by explosion and transitivity. By the deduction theorem, $\Sigma_0 \cup \Xi \vdash A \supset B$.
- If $\Sigma_0 \cup \Xi \cup \{A\} \nvdash \bot$ then by Lemma 9, $\Xi \in \mathrm{MCS}(\langle \Sigma_0 \cup \{A\}, \Sigma_d \rangle)$ and thus $\Sigma_0 \cup \Xi \vdash B$. By monotonicity, $\Sigma_0 \cup \Xi \cup \{A\} \vdash B$. By the deduction theorem, $\Sigma_0 \cup \Xi \vdash A \supset B$.

This is an interesting asymmetry between the two consequence relations: while the universal one validates the Deduction Theorem (in the factual premises), the free one doesn’t.

Should we expect the same behavior for the Deduction Theorem in the defeasible premises?

Well, here’s the surprise …

We now suppose the Deduction Theorem holds for the Base Logic **L** and prove:

The deduction theorem in the defeasible premises

$\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_{\text{free}} B$ implies $\langle \Sigma_0, \Sigma_d \rangle \vdash_{\text{free}} A \supset B$.

Proof.

- Suppose $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_{\text{free}} B$ and hence $\Sigma_0 \cup \bigcap \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle) \vdash B$.
- It is easy to see that

$$\bigcap \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle) \subseteq \bigcap \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \cup \{A\} \qquad (4)$$

- By monotonicity, $\Sigma_0 \cup \bigcap \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \cup \{A\} \vdash B$ and by the deduction theorem, $\Sigma_0 \cup \bigcap \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) \vdash A \supset B$.

□

Exercise

Exercise

Prove (4) in the proof above.

Deduction Thm for defeasible premises, Universal Consequences

We now suppose the Deduction Theorem holds for the Base Logic **L** and prove:

The deduction theorem in the defeasible premises

$\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$ implies $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A \supset B$.

Proof.

- Suppose $\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle \vdash_\forall B$ and let $\Xi \in \mathrm{MCS}(\Sigma_0, \Sigma_d)$.
- We have two case: (1) $\Xi \cup \{A\} \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \cup \{A\} \rangle)$ or (2) $\Xi \in \mathrm{MCS}(\Sigma_0, \Sigma_d \cup \{A\})$ and $\Xi \cup \Sigma_0 \cup \{A\} \vdash \bot$ and thus $\Xi \cup \Sigma_0 \cup \{A\} \vdash B$ (by $\bot$-explosion (!) and transitivity).
    1. Then by the supposition $\Sigma_0 \cup \Xi \cup \{A\} \vdash B$ and hence $\Sigma_0 \cup \Xi \vdash A \supset B$ by the deduction theorem.
    2. $\Sigma_0 \cup \Xi \vdash A \supset B$ by the deduction theorem.
- Thus, $\Sigma_0 \cup \Xi \vdash A \supset B$ and $\langle \Sigma_0, \Sigma_d \rangle \vdash_\forall A \supset B$.

Some Meta-Theory

Summing up

OK, this gets a bit exhausting … maybe time to sum this up!

An overview

	$\vdash_{\text{free}}$	$\vdash_{\forall}$
Ded.Thm. (factual)		✓
Ded.Thm. (defeasible)	✓	✓
Res.Thm (factual)		
Res.Thm (defeasible)		
CM (factual)	✓	✓
CM (defeasible)	✓	✓
Cut (factual)	✓	✓
Cut (defeasible)	✓	✓
RM (factual)		
RM (defeasible)		

From MCSs to Selection Semantics

From MCSs to Selection Semantics

Some basic definitions and suppositions

… concerning L

We now suppose that **L** has an adequate semantics with no inconsistent models.

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- $\Sigma \vdash A$ iff $\Sigma \Vdash A$ (where $\Sigma \Vdash A$ iff for all $M \in \mathcal{M}(\Sigma), M \models A$)

We now suppose that **L** has an adequate semantics with no inconsistent models.

- $\Sigma \vdash A$ iff $\Sigma \Vdash A$ (where $\Sigma \Vdash A$ iff for all $M \in \mathcal{M}(\Sigma), M \models A$)
- $\mathcal{M}(\{\bot\}) = \emptyset$

The defeasible part of a model

Definition 10

Given $\boldsymbol{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$,

- where $M \in \mathcal{M}(\Sigma_0)$, its defeasible part $d(M)$ is $\{A \in \Sigma_d \mid M \models A\}$

The defeasible part of a model

Definition 10

Given $\boldsymbol{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$,

- where $M \in \mathcal{M}(\Sigma_0)$, its defeasible part $d(M)$ is $\{A \in \Sigma_d \mid M \models A\}$
- $\mathcal{M}^m(\boldsymbol{\Sigma}) = \{M \in \mathcal{M}(\Sigma_0) \mid$ there are no $M' \in \mathcal{M}(\Sigma_0)$ for which $d(M) \subset d(M')\}$

Two central lemmas

Lemma 11

$\Sigma_0 \cup \Xi$ *is consistent iff* $\mathcal{M}(\Sigma_0 \cup \Xi) \neq \emptyset$

Proof.

- $\Sigma_0 \cup \Xi$ is inconsistent iff
- $\Sigma_0 \cup \Xi \vdash \bot$ iff
- $\Sigma_0 \cup \Xi \Vdash \bot$ iff
- $\mathcal{M}(\Sigma_0 \cup \Xi) = \emptyset$

□

Two central lemmas (cont.)

Lemma 12

$\mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle) = \{d(M) \mid M \in \mathcal{M}^m(\langle \Sigma_0, \Sigma_d \rangle)\}$

Proof.

- $(\subseteq)$ Suppose $(\dagger)$ $\Xi \in \mathrm{MCS}(\langle \Sigma_0, \Sigma_d \rangle)$.
- Assume there is no $M \in \mathcal{M}(\Sigma_0)$ for which $d(M) \supseteq \Xi$.
- Hence, $\mathcal{M}(\Sigma_0 \cup \Xi) = \emptyset$ and thus $\Sigma_0 \cup \Xi \Vdash \bot$.
- Since then $\Sigma_0 \cup \Xi \vdash \bot$: contradiction to $(\dagger)$.
- Assume there is a $M \in \mathcal{M}(\Sigma_0)$ for which $d(M) \supset \Xi$.
- Then, $\mathcal{M}(\Sigma_0 \cup d(M)) \neq \emptyset$ and hence $\Sigma_0 \cup d(M) \nVdash \bot$.
- Since then $\Sigma_0 \cup d(M) \nvdash \bot$: contradiction to $(\dagger)$.

□

Exercise

Prove the other direction of Lemma 12.

From MCSs to Selection Semantics

A selection semantics for the universal consequence

Definition 13

Where $\boldsymbol{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\boldsymbol{\Sigma} \Vdash_m A$ iff for all $M \in \mathcal{M}^m(\boldsymbol{\Sigma})$, $M \models A$.

Theorem 14

Where $\boldsymbol{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\boldsymbol{\Sigma} \vdash_\forall A$ *iff* $\boldsymbol{\Sigma} \Vdash_m A$

Proof.

- $\boldsymbol{\Sigma} \vdash_\forall A$, iff
- for all $\Xi \in \mathrm{MCS}(\boldsymbol{\Sigma})$, $\Sigma_0 \cup \Xi \vdash A$, iff
- for all $\Xi \in \mathrm{MCS}(\boldsymbol{\Sigma})$, $\Sigma_0 \cup \Xi \Vdash A$, iff
- for all $\Xi \in \mathrm{MCS}(\boldsymbol{\Sigma})$ and for all $M \in \mathcal{M}(\Sigma_0 \cup \Xi)$, $M \models A$, iff
- for all $M \in \mathcal{M}^m(\boldsymbol{\Sigma})$, $M \models A$, (here we use Lemma 12) iff
- $\boldsymbol{\Sigma} \Vdash_m A$.

□

From MCSs to Selection Semantics

A selection semantics for the free consequence

The semantic selection

Definition 15

Where $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$,

- $\mathcal{M}^f(\mathbf{\Sigma}) = \{M \in \mathcal{M}(\Sigma_0) \mid d(M) \supseteq \bigcap\{d(M') \mid M' \in \mathcal{M}^m(\mathbf{\Sigma})\}\}$.
- $\mathbf{\Sigma} \Vdash_f A$ iff for all $M \in \mathcal{M}^f(\mathbf{\Sigma})$, $M \models A$

Soundness and Completeness

Theorem 16

Where $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\mathbf{\Sigma} \vdash_{\mathrm{free}} A$ *iff* $\mathbf{\Sigma} \Vdash_f A$

Proof.

- $\mathbf{\Sigma} \vdash_{\mathrm{free}} A$, iff

Theorem 16

Where $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\mathbf{\Sigma} \vdash_{\mathrm{free}} A$ *iff* $\mathbf{\Sigma} \Vdash_f A$

Proof.

- $\mathbf{\Sigma} \vdash_{\mathrm{free}} A$, iff
- $\Sigma_0 \cup \mathrm{Free}(\mathbf{\Sigma}) \vdash A$, iff

Soundness and Completeness

Theorem 16

Where $\boldsymbol{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\boldsymbol{\Sigma} \vdash_{\mathrm{free}} A$ *iff* $\boldsymbol{\Sigma} \Vdash_f A$

Proof.

- $\boldsymbol{\Sigma} \vdash_{\mathrm{free}} A$, iff
- $\Sigma_0 \cup \mathrm{Free}(\boldsymbol{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\boldsymbol{\Sigma}) \vdash A$, iff

Soundness and Completeness

Theorem 16

Where $\boldsymbol{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\boldsymbol{\Sigma} \vdash_{\mathrm{free}} A$ *iff* $\boldsymbol{\Sigma} \Vdash_f A$

Proof.

- $\boldsymbol{\Sigma} \vdash_{\mathrm{free}} A$, iff
- $\Sigma_0 \cup \mathrm{Free}(\boldsymbol{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\boldsymbol{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\boldsymbol{\Sigma}) \Vdash A$, iff

Soundness and Completeness

Theorem 16

Where $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\mathbf{\Sigma} \vdash_{\mathrm{free}} A$ *iff* $\mathbf{\Sigma} \Vdash_f A$

Proof.

- $\mathbf{\Sigma} \vdash_{\mathrm{free}} A$, iff
- $\Sigma_0 \cup \mathrm{Free}(\mathbf{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\mathbf{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\mathbf{\Sigma}) \Vdash A$, iff
- for all $M \in \mathcal{M}(\Sigma_0 \cup \bigcap \mathrm{MCS}(\mathbf{\Sigma}))$, $M \models A$, iff

Soundness and Completeness

Theorem 16

Where $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\mathbf{\Sigma} \vdash_{\text{free}} A$ *iff* $\mathbf{\Sigma} \Vdash_f A$

Proof.

- $\mathbf{\Sigma} \vdash_{\text{free}} A$, iff
- $\Sigma_0 \cup \mathrm{Free}(\mathbf{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\mathbf{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\mathbf{\Sigma}) \Vdash A$, iff
- for all $M \in \mathcal{M}(\Sigma_0 \cup \bigcap \mathrm{MCS}(\mathbf{\Sigma}))$, $M \models A$, iff
- for all $M \in \mathcal{M}(\Sigma_0)$ such that $d(M) \supseteq \bigcap \mathrm{MCS}(\mathbf{\Sigma})$, $M \models A$, iff

Theorem 16

Where $\mathbf{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\mathbf{\Sigma} \vdash_{\mathrm{free}} A$ *iff* $\mathbf{\Sigma} \Vdash_f A$

Proof.

- $\mathbf{\Sigma} \vdash_{\mathrm{free}} A$, iff
- $\Sigma_0 \cup \mathrm{Free}(\mathbf{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\mathbf{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\mathbf{\Sigma}) \Vdash A$, iff
- for all $M \in \mathcal{M}(\Sigma_0 \cup \bigcap \mathrm{MCS}(\mathbf{\Sigma}))$, $M \models A$, iff
- for all $M \in \mathcal{M}(\Sigma_0)$ such that $d(M) \supseteq \bigcap \mathrm{MCS}(\mathbf{\Sigma})$, $M \models A$, iff
- for all $M \in \mathcal{M}(\Sigma_0)$ such that $d(M) \supseteq \bigcap \{d(M') \mid M' \in \mathcal{M}^m(\mathbf{\Sigma})\}$, $M \models A$, iff

Theorem 16

Where $\boldsymbol{\Sigma} = \langle \Sigma_0, \Sigma_d \rangle$, $\boldsymbol{\Sigma} \vdash_{\text{free}} A$ *iff* $\boldsymbol{\Sigma} \Vdash_f A$

Proof.

- $\boldsymbol{\Sigma} \vdash_{\text{free}} A$, iff
- $\Sigma_0 \cup \mathrm{Free}(\boldsymbol{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\boldsymbol{\Sigma}) \vdash A$, iff
- $\Sigma_0 \cup \bigcap \mathrm{MCS}(\boldsymbol{\Sigma}) \Vdash A$, iff
- for all $M \in \mathcal{M}(\Sigma_0 \cup \bigcap \mathrm{MCS}(\boldsymbol{\Sigma}))$, $M \models A$, iff
- for all $M \in \mathcal{M}(\Sigma_0)$ such that $d(M) \supseteq \bigcap \mathrm{MCS}(\boldsymbol{\Sigma})$, $M \models A$, iff
- for all $M \in \mathcal{M}(\Sigma_0)$ such that $d(M) \supseteq \bigcap \{d(M') \mid M' \in \mathcal{M}^m(\boldsymbol{\Sigma})\}$, $M \models A$, iff
- $\boldsymbol{\Sigma} \Vdash_f A$.

Constrained Input/Output Logic

Constrained Input/Output Logic

Input/Output Logic (Derivational Perspective)

The setup

- We have a set of input formulas Σ_0 in some propositional language $\mathcal{L}$
- And a set of Input/Output pairs Σ_{io} of the form (A, B) where $A, B \in \mathcal{L}$
- Think of them as conditionals or rules to which Modus Ponens should be applied

Input/Output Logic

Each Input/Output logic $\mathbf{L}(\mathcal{R})$ is based on a base logic $\mathbf{L}$ and comes with a set of rules $\mathcal{R}$ on the elements in Σ_{io}. Here are some examples:

- WO: If $A \vdash C$, then from (B, A) derive (B, C).
- SI: If $C \vdash A$, then from (A, B) derive (C, B).
- AND: From (A, B) and (A, C) derive $(A, B \wedge C)$.
- OR: From (B, A) and (C, A) derive $(B \vee C, A)$.
- CT: From (A, B) and $(A \wedge B, C)$ derive (A, C).
- ID: (A, A)
- T: $(\top, \top)$

Definition 17

Where $\Sigma_{io}^{\mathcal{R}}$ is the set of all (A, B) derivable from Σ_0 via the rules in $\mathcal{R}$,
$out_{\mathbf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io} \rangle) = \{A \mid \exists (B, A) \in \Sigma_{io}^{\mathcal{R}}, \exists B \in Cn(\Sigma_0)\}$.

Constrained Input/Output Logic

Constrained Input/Output Logic

Problem: conflicts

Example

- Suppose $\Sigma_0 = \{p, q\}$,
- $\Sigma_{io} = \{(p, \neg q)\}$,
- and we have (at least) the rules ID, AGG, WO, SI in $\mathcal{R}$
    - via ID we have (q, q) and via SI $(p \wedge q, q)$
    - via SI we have $(p \wedge q, \neg q)$.
    - via AGG, $(p \wedge q, q \wedge \neg q)$ and via WO $(p \wedge q, A)$ for any A
- Then $out_{CL(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io} \rangle)$ is trivial.

Solution: "go MCS"

The idea is analogous to what we had before:

- premises in Σ_0 are strict
- IO-pairs in Σ_{io} are defeasible
- $\Xi_{io} \subseteq \Sigma_{io}$ is a maximal consistent subset of $\langle \Sigma_0, \Sigma_{io} \rangle$ iff
  1. $out_{\mathsf{L}(\mathcal{R})}(\Sigma_0, \Xi_{io})$ is consistent
  2. there is no $\Xi'_{io} \subseteq \Sigma_{io}$ such that $out_{\mathsf{L}(\mathcal{R})}(\Sigma_0, \Xi'_{io})$ is consistent and $\Xi_{io} \subset \Xi'_{io}$.
- We write $\Xi_{io} \in \mathrm{MCS}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io} \rangle)$.

Definition 18

- $out^{\forall}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io} \rangle) = \bigcap \{ out_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Xi_{io} \rangle) \mid \Xi_{io} \in \mathrm{MCS}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io} \rangle) \}$.
- $out^{\exists}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io} \rangle) = \bigcup \{ out_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Xi_{io} \rangle) \mid \Xi_{io} \in \mathrm{MCS}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io} \rangle) \}$.

Examples!

First our previous example where $\Sigma_0 = \{p, q\}$ and $\Sigma_{io} = \{(p, \neg q)\}$ and we had (at least) the rules ID, AGG, WO, SI in $\mathcal{R}$.

- As we saw, $out_{CL(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io} \rangle)$ is trivial.
- Thus, the only member of $\mathrm{MCS}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io} \rangle)$ is $\emptyset$.
- This means that $out^{\forall}_{\mathsf{L}(\mathcal{R})}(\Sigma_0, \Sigma_{io}) = out_{\mathsf{L}(\mathcal{R})}(\Sigma_0, \emptyset)$.
- Which of the following are consequences?
  1. p
  2. q
  3. $\top$
  4. $p \wedge q$

Another example

- Let $\Sigma_0 = \{p, s\}$ and
- $\Sigma_{io} = \{(p, q), (s, \neg q), (p, r)\}$
- Let $\mathcal{R}$ contain WO, SI, AGG.
- our $\mathrm{MCS}(\langle \Sigma_0, \Sigma_{io} \rangle)$ has two members:
    1. $\{(p, q), (p, r)\}$
    2. $\{(s, \neg q), (p, r)\}$
- This means e.g., $r \in out^{\forall}_{\mathcal{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io} \rangle)$.

But where are the constraints in “Constrained Input/Output Logic”?

The full picture

The idea is analogous to what we had before:

- premises in Σ_0 are strict
- IO-pairs in Σ_{io} are defeasible
- Σ_c, a set of formulas in $\mathcal{L}$, represent constraints
- $\Xi_{io} \subseteq \Sigma_{io}$ is a maximal consistent subset of $\langle \Sigma_0, \Sigma_{io}, \Sigma_c \rangle$ iff
  1. $out_{\mathsf{L}(\mathcal{R})}(\Sigma_0, \Xi_{io}) \cup \Sigma_c$ is consistent
  2. there is no $\Xi'_{io} \subseteq \Sigma_{io}$ such that $out_{\mathsf{L}(\mathcal{R})}(\Sigma_0, \Xi'_{io}) \cup \Sigma_c$ is consistent and $\Xi_{io} \subset \Xi'_{io}$.
- We write $\Xi_{io} \in \mathrm{MCS}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io}, \Sigma_c \rangle)$.

Definition 19

- $out^{\forall}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io}, \Sigma_c \rangle) = \bigcap \{ out_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Xi_{io} \rangle) \mid \Xi_{io} \in \mathrm{MCS}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io}, \Sigma_c \rangle) \}$.
- $out^{\exists}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io}, \Sigma_c \rangle) = \bigcup \{ out_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Xi_{io} \rangle) \mid \Xi_{io} \in \mathrm{MCS}_{\mathsf{L}(\mathcal{R})}(\langle \Sigma_0, \Sigma_{io}, \Sigma_c \rangle) \}$.

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