A logical approach to Isomorphism Testing and Constraint Satisfaction

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ESSLLI 2016, 15–19 August

Part 2:
Isomorphism Testing by Color Refinement and $\mathrm{FO}^2_{\#}$

Outline

1. Graph Isomorphism Problem
2. Color Refinement Algorithm
3. Relation to $\mathrm{FO}^2_{\#}$
4. References

Outline

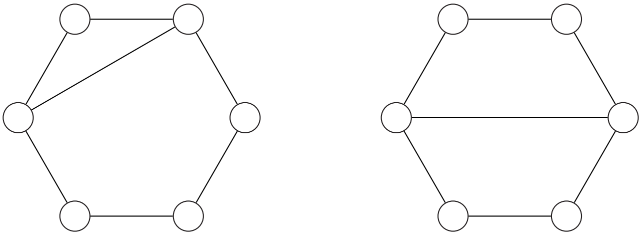
Graph Isomorphism Problem

Are two given graphs G and H isomorphic?

- the best algorithm takes time $n^{\log^c n}$ [Babai 2015]
- in NP, but not NP-complete unless the polynomial-time hierarchy collapses [Schöning 1988, Boppana, Håstad, Zachos 1987]
- polynomial time algorithms are known only in particular cases, e.g., for
  - bounded genus [Filotti, Mayer 1980; Miller 1980]
  - bounded degree [Luks 1982]
  - more generally,
    classes excluding a topological minor [Grohe, Marx 2012]
  - interval graphs [Lueker, Booth 1979]
- in some cases, even logspace algorithms:
  - bounded genus [Elberfeld, Kawarabayashi 2014]
  - bounded treewidth [Elberfeld, Schweitzer 2016]
  - interval graphs [Köbler, Kuhnert, Laubner, V. 2011]

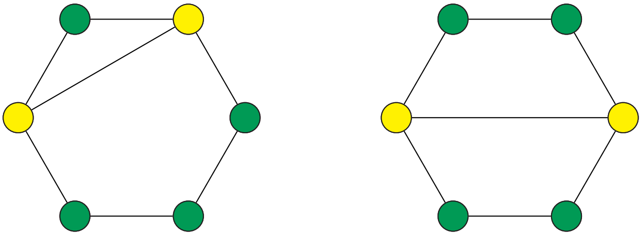
Outline

Color refinement algorithm: An example



Start with the monochromatic coloring.

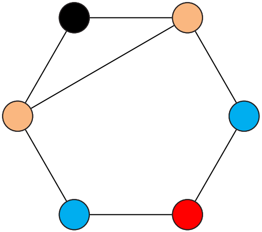
Color refinement algorithm: An example



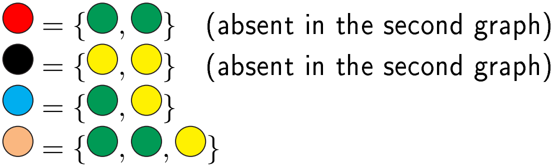
New color of a vertex = old colors of all neighbours.

● $= \{\bigcirc, \bigcirc\}$

● $= \{\bigcirc, \bigcirc, \bigcirc\}$



Next refinement.



Color refinement algorithm: Formal definition

$$\begin{aligned} C^1(v) &= \deg v \\ C^{i+1}(v) &= \{\!\{\, C^i(u) \,\}\!\}_{u \in N(v)} \end{aligned}$$

Exercise

If ϕ is an isomorphism from G to H, then $C^i(v) = C^i(\phi(v))$.

Therefore,

$$G \cong H \implies \{\!\{\, C^i(v) \,\}\!\}_{v \in V(G)} = \{\!\{\, C^i(v) \,\}\!\}_{v \in V(H)}$$

Color Refinement accepts G and H as isomorphic

iff

the equality is true for all i.

- The output "non-isomorphic" is always true.
- The output "isomorphic" can be wrong.

How many refinement steps are needed on n-vertex graphs?

Exercise

$C^{i+1}(v) = C^{i+1}(v') \implies C^i(v) = C^i(v')$.

Regard C^i as a coloring of the graph $F = G + H$.

Let $\mathcal{P}^i$ be the partition of $V(F) = V(G) \cup V(H)$ according to C^i.

$\mathcal{P}^{i+1}$ is a refinement of $\mathcal{P}^i$ (by Exercise) $\implies$

$\mathcal{P}^s = \mathcal{P}^{s+1}$ for some $s < 2n \implies$

For any $X, Y \in \mathcal{P}^s$, the induced subgraph $F[X]$ is regular and the induced bipartite subgraph $F[X, Y]$ is bi-regular. $\implies$

$\mathcal{P}^{s+1} = \mathcal{P}^{s+2} = \ldots$ (partition stabilization) $\implies$

For any $v, v' \in V(G) \cup V(H)$, if $C^s(v) = C^s(v')$ then $C^i(v) = C^i(v')$ for all $i \geq s$. $\implies$

CR distinguishes G and H either in $s < 2n$ steps or never.

How many refinement steps are needed on n-vertex graphs?

In fact, n refinement steps are enough.

Indeed, if $\mathcal{P}^{i+1}$ is a proper refinement of $\mathcal{P}^i$ and

$$\{\!\{\, C^{i+1}(v) \,\}\!\}_{v \in V(G)} = \{\!\{\, C^{i+1}(v) \,\}\!\}_{v \in V(H)},$$

then $\mathcal{P}^{i+1}$ is a proper refinement of $\mathcal{P}^i$ on both $V(G)$ and $V(H)$.

Color refinement algorithm: Implementation details

The length of $C^i(v)$ grows exponentially as i increases.

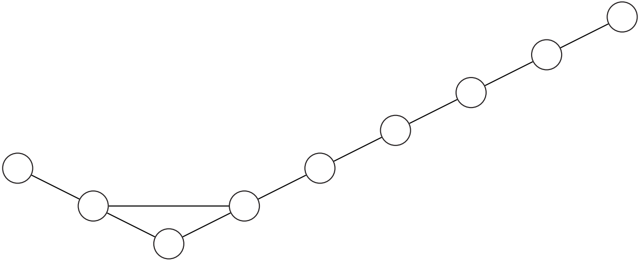
Solution: Enumerate (rename) the colors lexicographically after each refinement round!

Graph Canonization Problem

Given: a graph G on the vertex set $\{1, \ldots, n\}$
Find: a permutation α_G of $\{1, \ldots, n\}$ such that

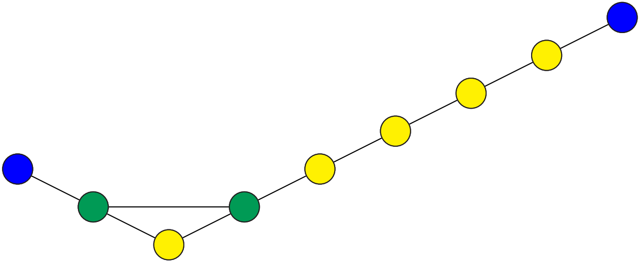
$$\alpha_G(G) = \alpha_H(H) \text{ whenever } G \cong H.$$

Color refinement algorithm: An example of canonization



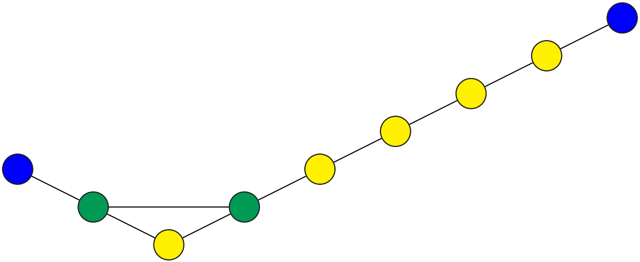
An input graph

Color refinement algorithm: An example of canonization



Initial coloring: $C^1(v) = \deg v$

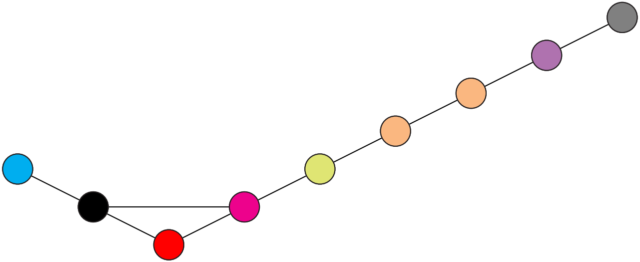
Color refinement algorithm: An example of canonization



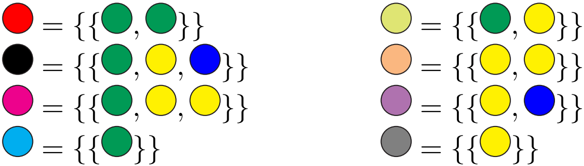
Initial coloring: $C^1(v) = \deg v$

A color refinement step: $C^{i+1}(v) = \{\!\!\{\, C^i(u) \,\}\!\!\}_{u \in N(v)}$

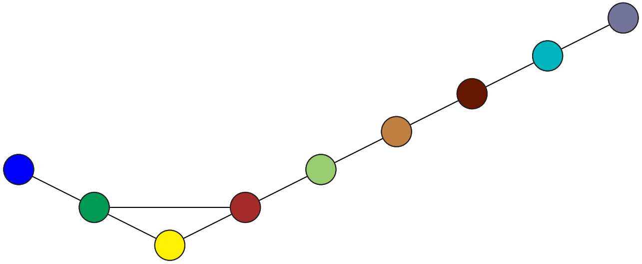
Color refinement algorithm: An example of canonization



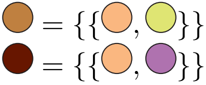
1st refinement step: $C^2(v) = \{\!\{\, C^1(u) \,\}\!\}_{u \in N(v)}$



Color refinement algorithm: An example of canonization



2nd refinement step: $C^3(v) = \{\!\!\{ \, C^2(u) \, \}\!\!\}_{u \in N(v)}$



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CR canonizes almost all graphs

Definition

We call G discrete if its stable partition consists of singletons.

Theorem (Babai, Erdős, Selkow 1980)

$G_{n,1/2}$ is discrete with high probability.

Proof-scheme

Let $m = o(\sqrt[4]{n/\log n})$ and U be the set of vertices with the m largest degrees. Then, with high probability,

- vertices in U have pairwise distinct degrees, cf. [Bollobás 1981]
- vertices not in U have pairwise distinct sets of neighbors in U, assuming also that $m > 3\log_2 n$.

Outline

Color Refinement and $\mathrm{FO}^2_{\#}$

Lemma (Immerman and Lander 1990)

For any possible C^i-color c there is $\Phi(x) \in \mathrm{FO}^2_{\#}$ such that

$$G, v \models \Phi(x) \quad \textit{iff} \quad C^i(v) = c$$

for every G and $v \in V(G)$.

Corollary

Discrete graphs are definable in $\mathrm{FO}^2_{\#}$.

Proof of the Immerman-Lander lemma

Base case $i = 1$.

$\deg v = d$ can be expressed by

$$\Phi(x) \stackrel{\text{def}}{=} \exists^{\geq d} y(y \sim x) \land \neg \exists^{\geq d+1} y(y \sim x)$$

(shorter: $\exists^{=d} y(y \sim x)$).

Induction step $i \mapsto i + 1$

Assumption: Each C^i-color c is definable by $\Phi_c(x)$.
Suppose $C^{i+1}(v) = c'$ iff v has s_1 neighbors u with $C^i(u) = c_1$, s_2 neighbors u with $C^i(u) = c_2$ and so on.
Then c' is definable by

$$\Phi_{c'}(x) \stackrel{\text{def}}{=} \bigwedge_j \exists^{=s_j} y \left(y \sim x \land \Phi_{c_j}(y)\right) \land \exists^{=\deg v} y \left(y \sim x\right).$$

Color Refinement and $\mathrm{FO}^2_{\#}$

Theorem (Immerman and Lander 1990)

The following two conditions are equivalent:

1. *G and H are indistinguishable by CR.*
2. *G and H are indistinguishable in $\mathrm{FO}^2_{\#}$.*

Example

C_6 is not definable in $\mathrm{FO}^2_{\#}$ because CR cannot distinguish it from $2C_3$.

Proof of the Immerman-Lander theorem

$\neg(1) \implies \neg(2)$ by the preceding lemma

$(1) \implies (2)$. The asumption (1) means that

$$\{\!\{ C^i(v) \}\!\}_{v\in V(G)} = \{\!\{ C^i(v) \}\!\}_{v\in V(H)} \text{ for all } i.$$

Let $\mathcal{P}^s$ be the stable partition of $V(G) \cup V(H)$. $\mathcal{P}^s$ consists of unions $Z \cup Z'$ such that $Z \subseteq V(G)$, $Z' \subseteq V(H)$, $|Z| = |Z'|$, and all vertices in Z have the same C^s-color as all vertices in Z'.

We design a winning strategy for Duplicator in the 2-pebble counting game on G and H.

1st round. If Spoiler marks a set of vertices $A \subseteq V(G)$, Duplicator responds with $B \subseteq V(H)$ such that

$$|A \cap Z| = |B \cap Z'| \text{ for every } Z \cup Z' \in \mathcal{P}^s$$

and ensures pebbling a pair of vertices $x \in X$ and $x' \in X'$ for some $X \cup X' \in \mathcal{P}^s$.

Proof of the Immerman-Lander theorem (cont'd)

For any $X \cup X' \in \mathcal{P}^s$ and $Y \cup Y' \in \mathcal{P}^s$

- $G[X]$ and $H[X']$ are regular graphs of the same degree;
- $G[X,Y]$ and $H[X',Y']$ are bi-regular graphs with the same degrees.

i-th round. Suppose that $x \in X$ and $x' \in X'$ are pebbled. If Spoiler marks $A \subseteq V(G)$, Duplicator responds with $B \subseteq V(H)$ such that

$$\begin{aligned} |A \cap (Z \cap N(x))| &= |B \cap (Z' \cap N(x'))|, \\ |A \cap (Z \setminus N(x))| &= |B \cap (Z' \setminus N(x'))| \end{aligned}$$

for every $Z \cup Z'$. Therewith she ensures pebbling a pair of vertices $y \in Y$ and $y' \in Y'$ for some $Y \cup Y' \in \mathcal{P}^s$ such that

$$y \in N(x) \iff y' \in N(x').$$

Conclusion

Color Refinement works correctly on G and every H

iff

G is definable in $\mathrm{FO}^2_{\#}$.

Conclusion

Color Refinement works correctly on G and every H
iff
G is definable in $\mathrm{FO}^2_{\#}$.

Question

Which graphs are definable in $\mathrm{FO}^2_{\#}$?

Outline

References

- N. Immerman and E. Lander. Describing graphs: A first-order approach to graph canonization. In *Complexity Theory Retrospective*, Springer, 1990.
- C. Berkholz, P.S. Bonsma, M. Grohe. Tight lower and upper bounds for the complexity of canonical colour refinement. ESA'13.