A logical approach to Isomorphism Testing and Constraint Satisfaction

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Part 4: $\mathrm{FO}^2_{\#}$ and linear-programming techniques.

Outline

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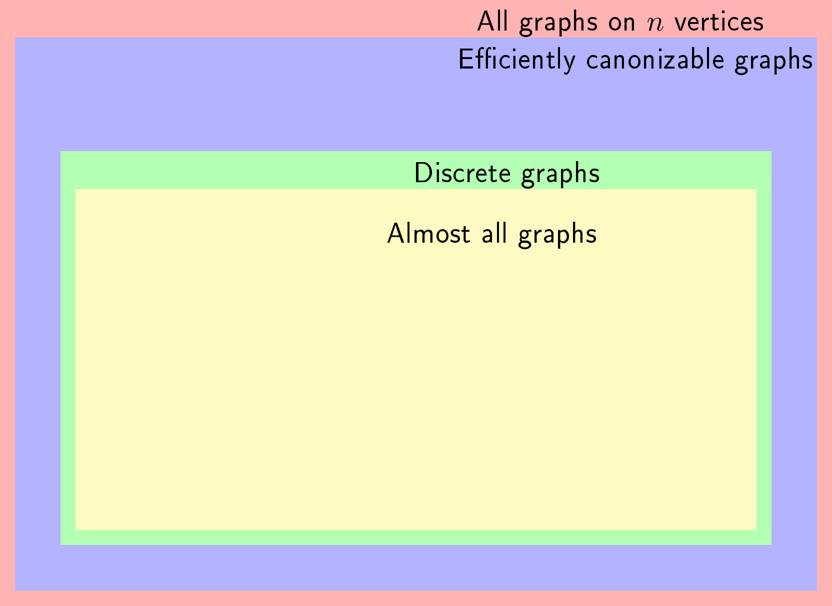
Canonizing almost all graphs



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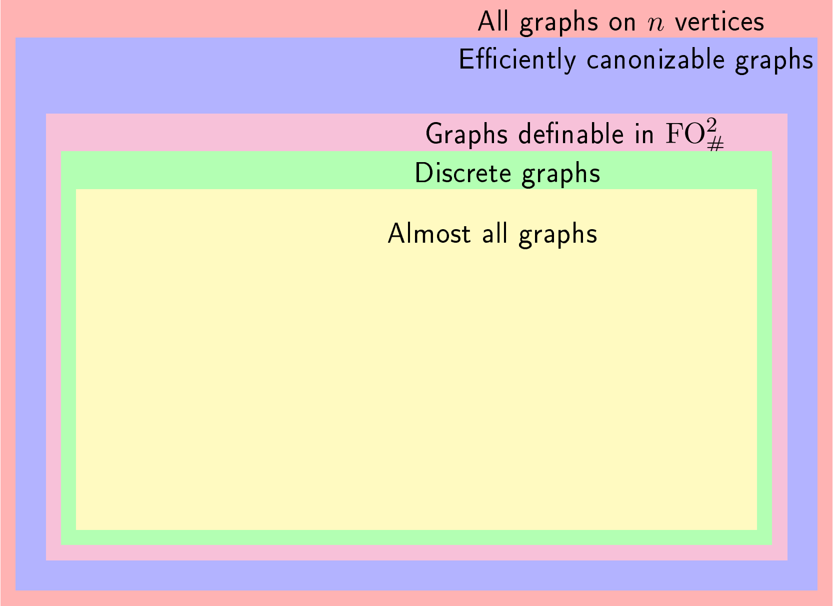

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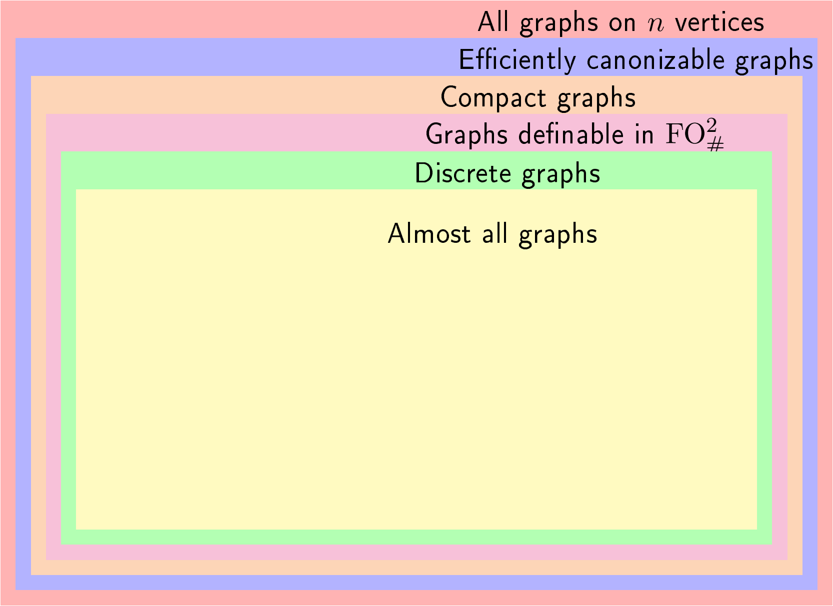

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Fractional isomorphism and equivalent concepts

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Theorem [Ramana, Scheinerman, Ullman 94; Immerman, Lander 90]

The following three conditions are equivalent:

- G and H are fractionally isomorphic,
- G and H are indistinguishable by Color Refinement,
- G and H are indistinguishable in $\mathrm{FO}^2_{\#}$.

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If G is known to be compact, then $G \cong H$ can be tested by computing an extreme point of $S(G, H) = \{ X \text{ - d.s.} : AX = XB \}$:

- If $G \cong H$, then all extreme points of $S(G, H)$ are integral;
- If $G \ncong H$, then $S(G, H)$ has no integral point.

Basic facts: Complete graphs

Complete graphs are compact.

Every $n \times n$ d.s. matrix is a fractional automorphism of K_n.
Indeed, let J and I be the all-ones and the identity matrices. Then

$$X \text{ is d.s.} \implies JX = XJ \implies (J-I)X = X(J-I) \implies X \in S(K_n).$$

Birkhoff's theorem

Every doubly stochastic matrix is a convex combination of permutation matrices.

Basic facts: Closure properties

G is compact iff its complement $\overline{G}$ is compact.

Proof:

$Aut(G) = Aut(\overline{G})$ and $S(G) = S(\overline{G})$. Indeed,

$$\begin{aligned} X \in S(\overline{G}) \quad &\iff \quad (J - I - A)X = X(J - I - A) \\ &\iff \quad AX = XA \iff X \in S(G), \end{aligned}$$

where A is the adjacency matrix of G.

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Lemma (Tinhofer 91)

If a connected graph G is compact, then the m-fold disjoint union mG is compact.

Example: The matching graph mK_2 and its complement $\overline{mK_2}$ are compact.

Further examples of compact graphs

Thus, the cycle graphs $C_3 = K_3$ and $C_4 = \overline{2K_2}$ are compact.

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Other examples:

- All cycles [Tinhofer 1986]
- All trees [Tinhofer 1986]
- Many regular graphs [Brualdi 88, Godsil 97, Wang and Li 05]

A negative example

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Lemma (Tinhofer 1991)

A regular compact graph is vertex-transitive.

Proof:

- Consider the $n \times n$ all-ones matrix J, where n is the number of vertices in G.
- If G is regular, then $\frac{1}{n}J \in S(G)$.
- If G is compact, then

$$\frac{1}{n}J = \sum_s \alpha_s P_s,$$

  a convex combination of permutation matrices from $Aut(G)$.
- Therefore, for all i and j there is s such that $[P_s]_{ij} = 1$.

Outline

$\mathrm{FO}^2_\#$-Definable ⊂ Compact

Theorem (Arvind, Köbler, Rattan, V. 2015)

All graphs definable in $\mathrm{FO}^2_\#$ *are compact.*

Proof-scheme

- The proof is based on our characterization of $\mathrm{FO}^2_{\#}$-definable graphs.

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Let G be a definable graph. If $X, Y \subseteq V(G)$ are vertex classes in the stable coloring of G (*cells*), then

- $G[X]$ is one of

$$K_s,\ \overline{K_s},\ mK_2,\ \overline{mK_2},\ \text{and } C_5.$$

- $G[X, Y]$ is one of

$$K_{s,t},\ \overline{K_{s+t}},\ sK_{1,t}(s \geq 2),\ \text{and its bipartite complement.}$$

Proof-scheme (cont'd)

- homogeneous (isotropic) links $G[X, Y]$ can be ignored;
- there is no cycle of non-homogeneous (anisotropic) links;
- each of the corresponding tree-like components of G can be considered separately;
- each such component contains at least one non-homogeneous cell X ($G[X] \cong mK_2, \overline{mK_2}$, or C_5);
- induction on the number of cells is possible because fractional automorphisms respect the stable partition [Ramana, Scheinerman, Ullman 1994];
- the base case is done by the compactness of K_s, $\overline{K_s}$, mK_2, $\overline{mK_2}$, and C_5.

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Tinhofer's canonization algorithm for compact graphs

Input: a graph G

1. Run Color Refinement on G till color stabilization.
2. If all color classes are singletons, terminate.
3. If there is a color class with 2 or more vertices, individualize one of them by assigning a new color (the lexicographically first unused one).
4. Goto Step 1.

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Theorem (Tinhofer 1991)

If an input graph G is compact, then the above algorithm produces a canonical labeling of G for any choice of vertices to be individualized.

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References

- G. Tinhofer. Graph isomorphism and theorems of Birkhoff type. Computing 36, 285–300 (1986).
- G. Tinhofer. A note on compact graphs. Discrete Applied Mathematics 30, 253–264 (1991).
- M.V. Ramana, E.R. Scheinerman, D. Ullman. Fractional isomorphism of graphs. Discrete Mathematics 132, 247–265 (1994).
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