A logical approach to Isomorphism Testing and Constraint Satisfaction

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Part 7: Alternation hierarchy of FO^2

Outline

1. The alternation function of FO^2
2. FO^2 is more succinct than FO^2_a
3. References

Outline

Definitions

G, H, ... will be binary structures (typically, vertex-colored graphs).

A sentence Φ distinguishes G from H if $G \models \Phi$ while $H \not\models \Phi$.

$$
\begin{aligned}
D^2(G,H) &= \text{the min quantifier depth of such } \Phi \in \mathrm{FO}^2. \\
A^2(G,H) &= \text{the min alternation depth of such } \Phi \in \mathrm{FO}^2 \\
& \quad \text{(only } \neg, \wedge, \vee \text{ are used, and } \neg \text{ always stays} \\
& \quad \text{in the front of relation symbols).}
\end{aligned}
$$

$$
\begin{aligned}
D^2(n) &= \max D^2(G,H), \\
A^2(n) &= \max A^2(G,H),
\end{aligned}
$$

where $\max$ is over n-element G and H distinguishable in FO^2.

Motivation

- FO^2_a denotes the fragment of FO^2 consisting of formulas of alternation depth at most a.

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- We will say that the quantifier alternation hierarchy of FO^2 collapses to its a-th level if
  - every property of graphs definable in FO^2 can also be defined in FO^2_a;
  - or, equivalently, every sentence in FO^2 has an equivalent sentence in FO^2_a.

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- If the alternation hierarchy collapses to the a-th level, then

$$A^2(n) \leq a.$$

- Thus, showing that

$$\lim_{n\to\infty} A^2(n) = \infty$$

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- If the alternation hierarchy collapses to the a-th level, then

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  is a way of proving that the hierarchy is strict.
- The rate of growth of $A^2(n)$ can be regarded as a quality of the strictness of the alternation hierarchy.

Bounds for $A^2(n)$ and $D^2(n)$

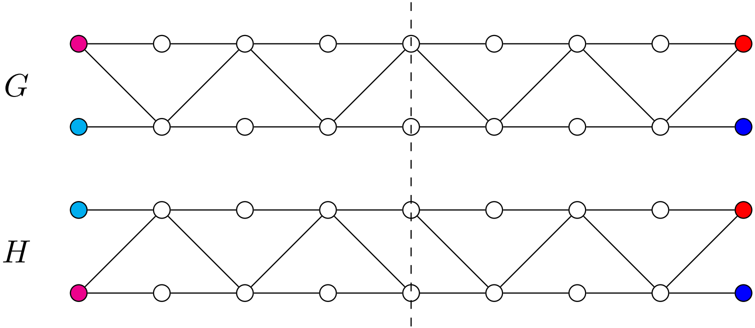
Theorem (Krebs, V. 2015)

$\frac{1}{8}n - 2 < A^2(n) \leq D^2(n) \leq n + 1$

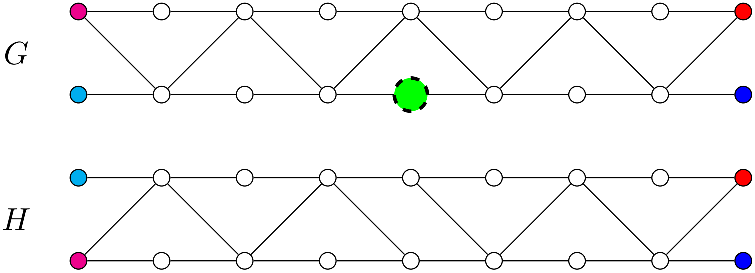
Remark

The upper bound due to Immerman and Lander 1990
(the color stabilization argument)

$A^2(n) > \frac{1}{8}\,n - 2$



Assumption: Spoiler pebbles along edges.



moves: ∃

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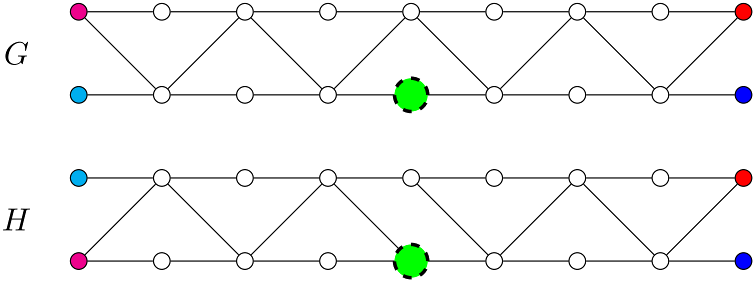

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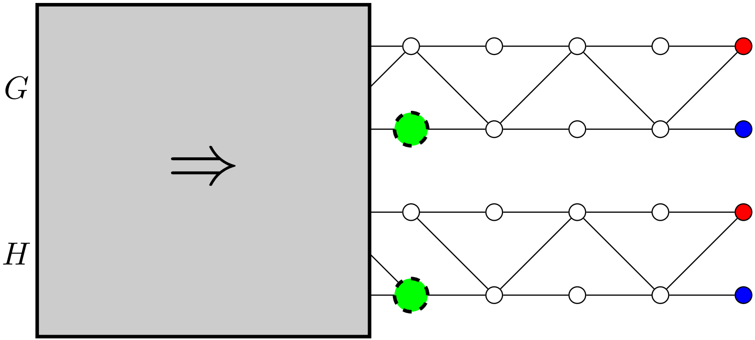

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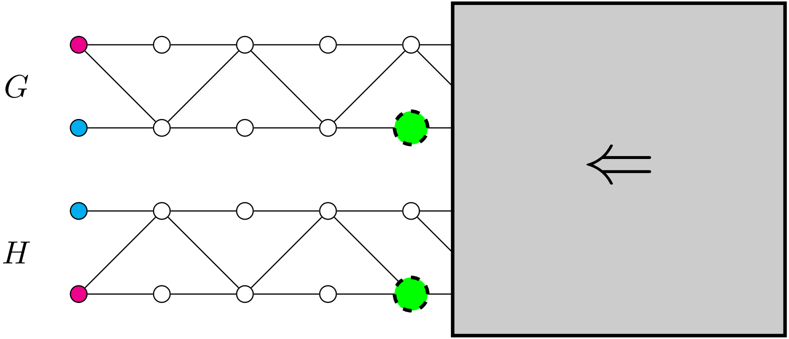

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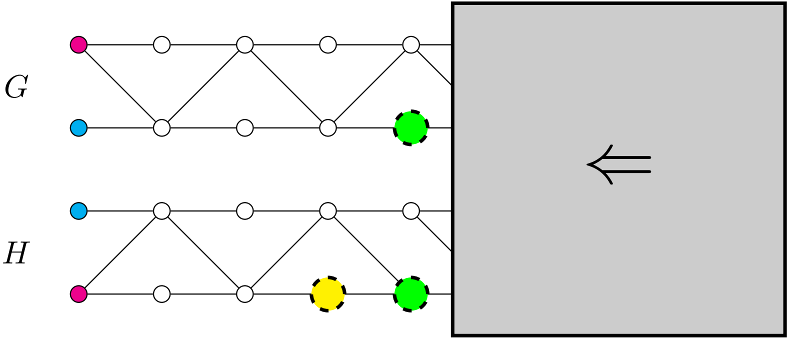

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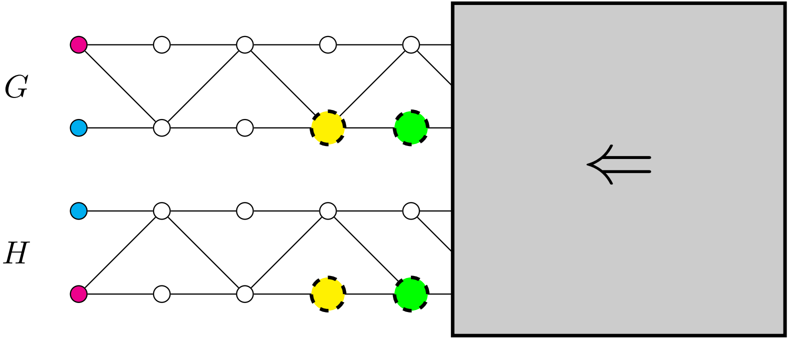

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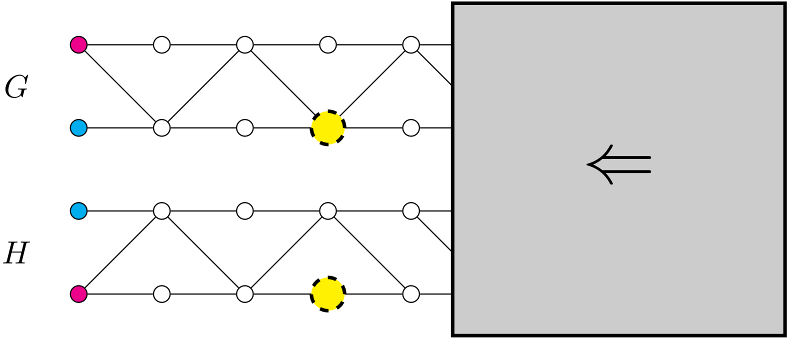

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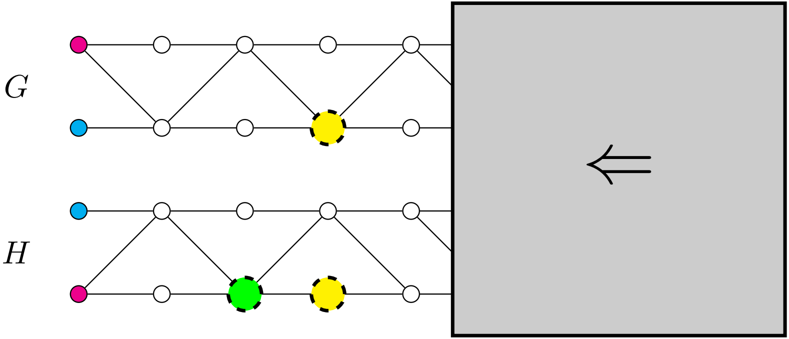

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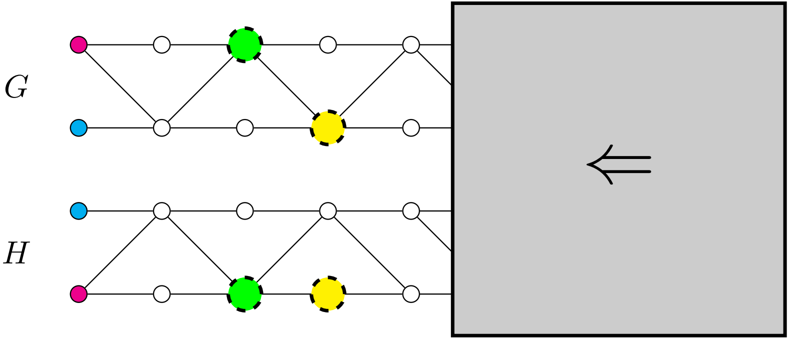

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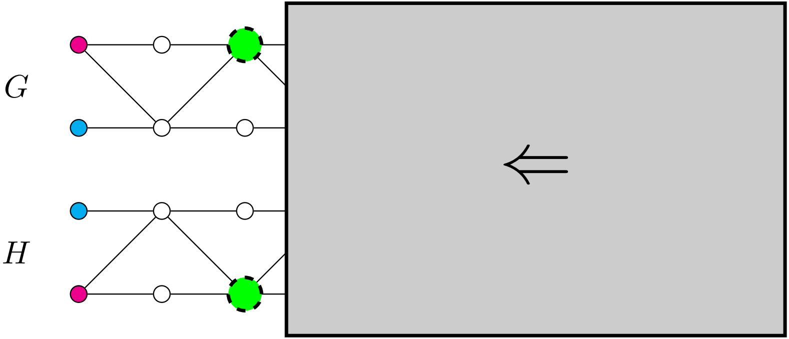

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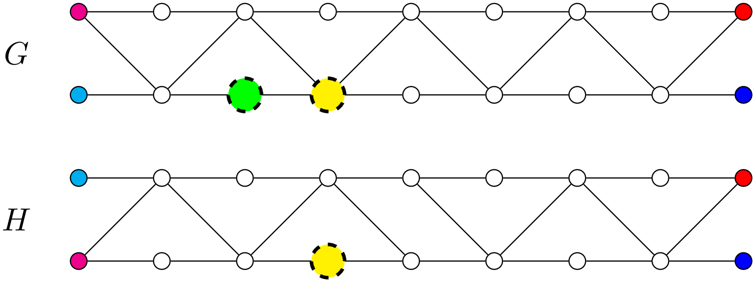

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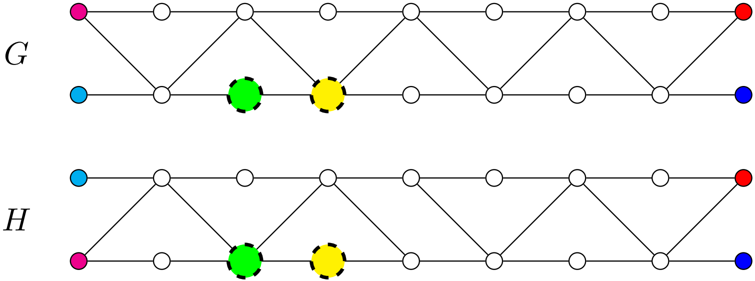

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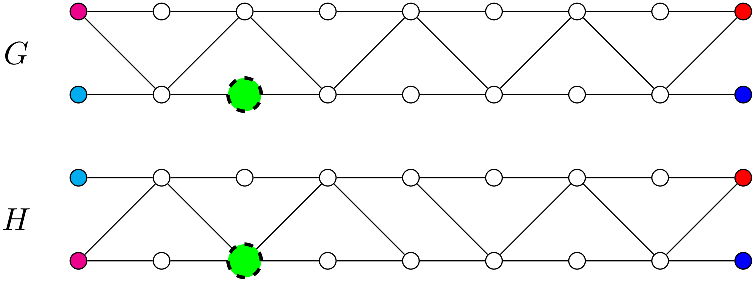

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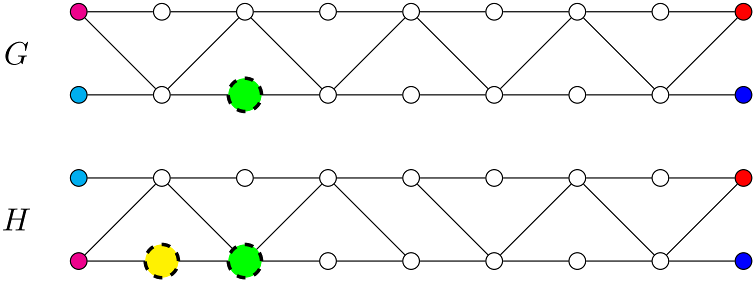

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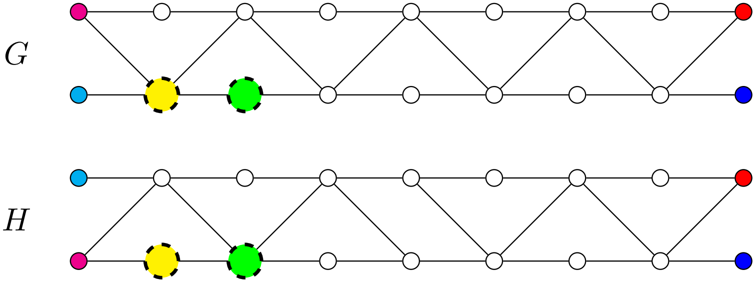

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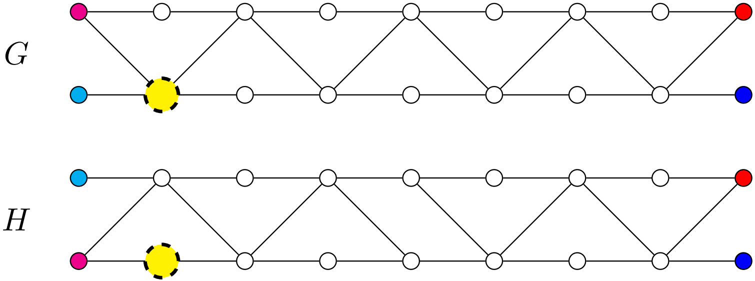

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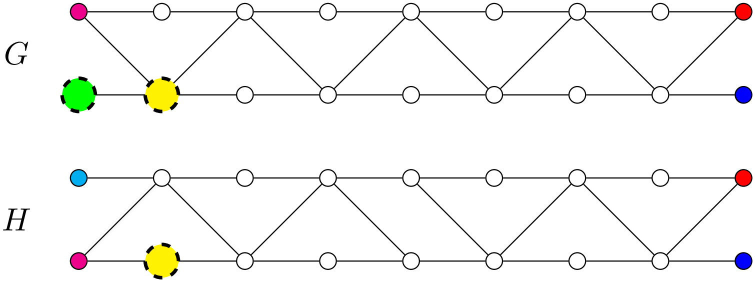

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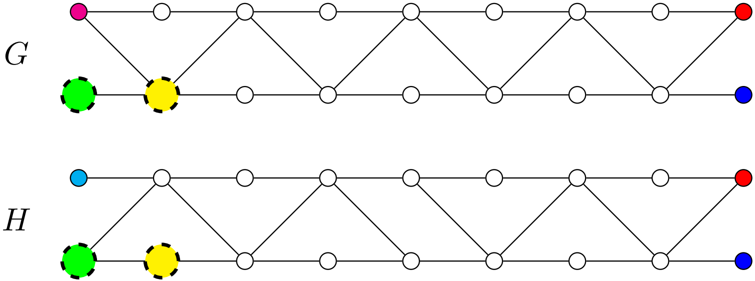
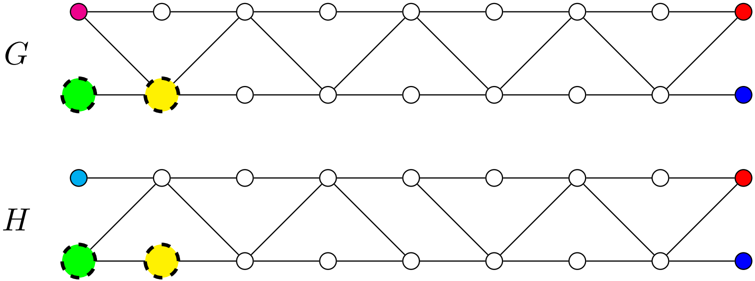


moves: $\exists\forall\exists\forall\exists$

$$A^2(n) > n/4 - 1$$

$A^2(n) > \frac{1}{8}\,n - 2$

~~Assumption: Spoiler pebbles along edges.~~

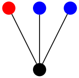


moves: $\exists\forall\exists\forall\exists$

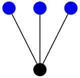
$A^2(n) > n/8 - 2$: Consider $2G$ and $2H$

$A^2(n) > \log_3 n - 2$ over trees (due to Chandra-Harel 1982)

G^1:



H^1:



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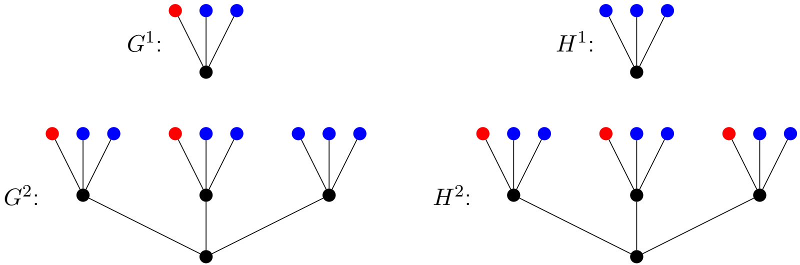

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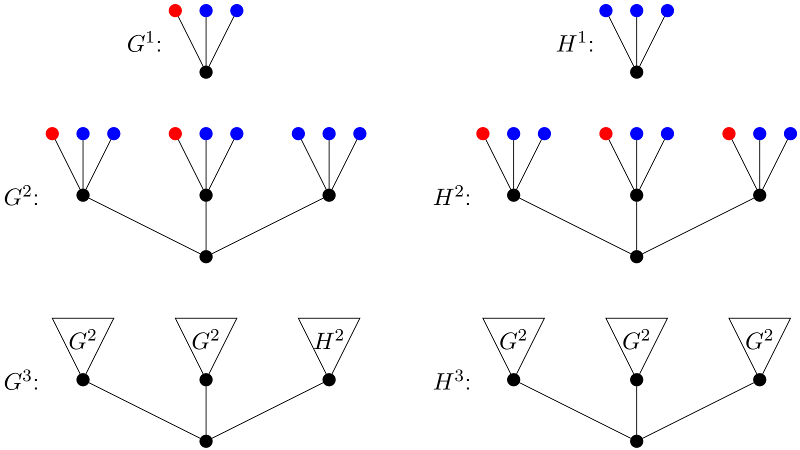

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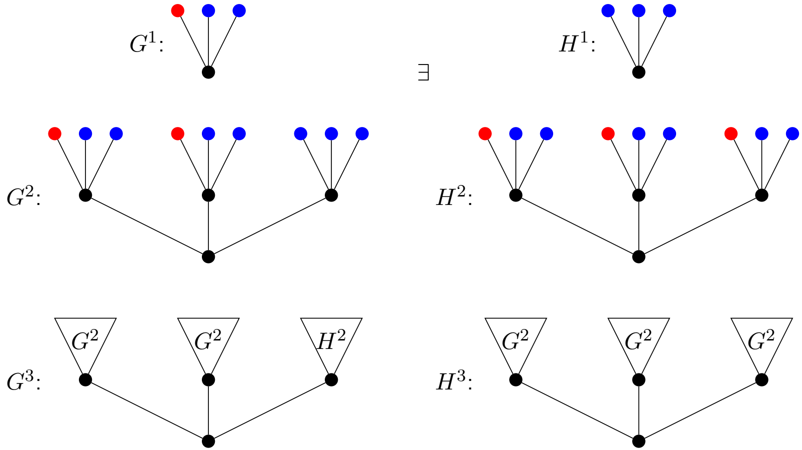

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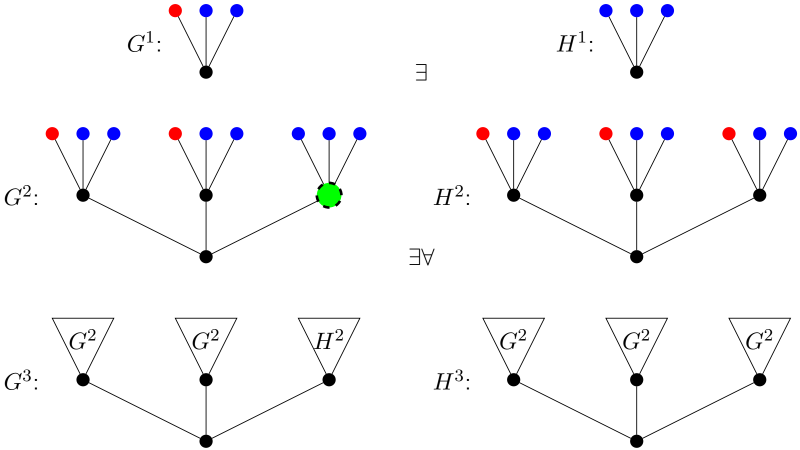

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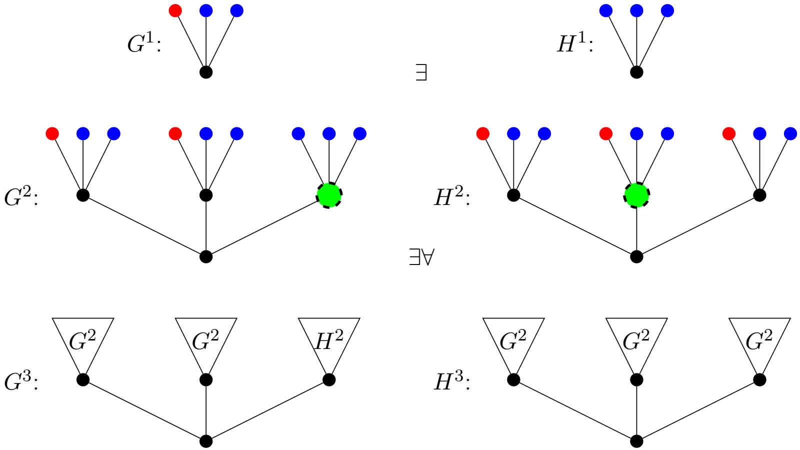

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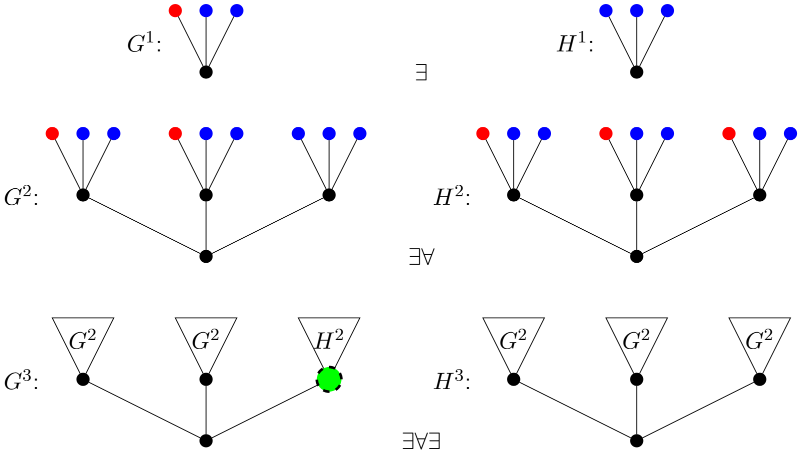

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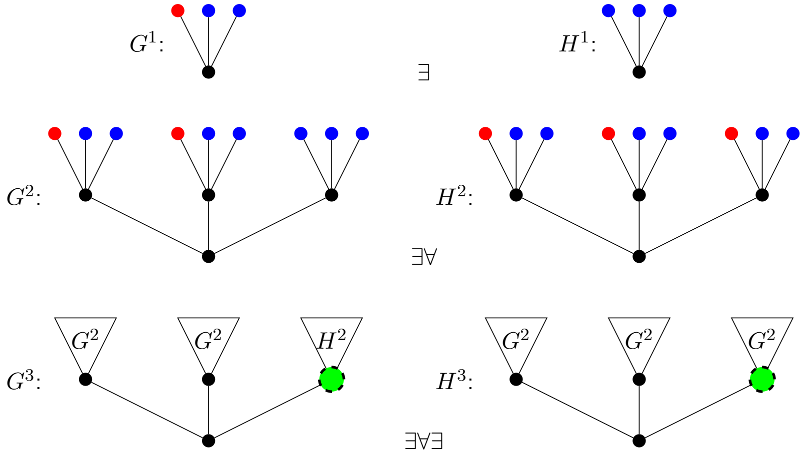

$A^2(n) > \log_3 n - 2$ over trees

Open problem

How tight is this lower bound?

Remark

If $k \geq 3$, then over trees

$$\log_{k+1} n - 2 < A^k(n) \leq D^k(n) < (k+3)\log_2 n.$$

Outline

FO^2 is more succinct than FO^2_a

Recall that $D^2(n) \leq n + 1$.

Let $D^2_a(n)$ be the analog of $D^2(n)$ for FO^2_a.

Theorem

$D^2_a(n) = \Omega(n^2)$ *for each* a.

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Remark

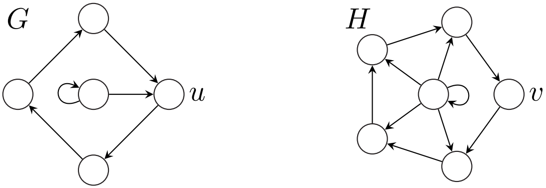
$D^2_a(n) \leq n^2 + 1$ for each a.
Proof-idea: If Spoiler is going to move one of the pebbles, the rest of the game is determined by the position $(u, v) \in V(G) \times V(H)$ of the other pebble pair. If the play is optimal and finite, the same position (u, v) never occurs twice.

Existential-positive FO^2 (recap)

Let $D^2_{\exists,+}(n)$ be the variant of $D^2(n)$ for $\mathrm{FO}^2_{\exists,+}$.

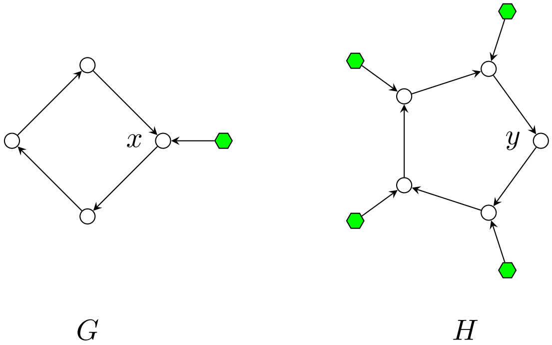
Theorem

$D^2_{\exists,+}(n) > \frac{1}{6}\,(n-10)^2$.



Note that $D^2_1(G,H) \leq 3$. This can be fixed.

Lifting it to FO_a^2



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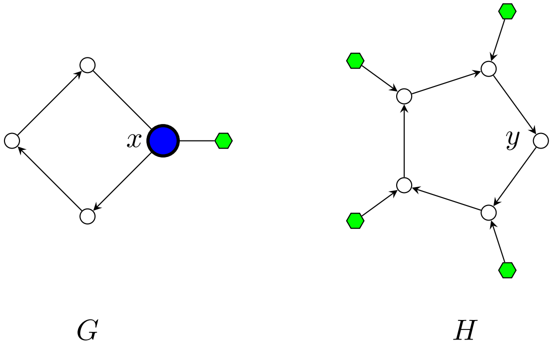

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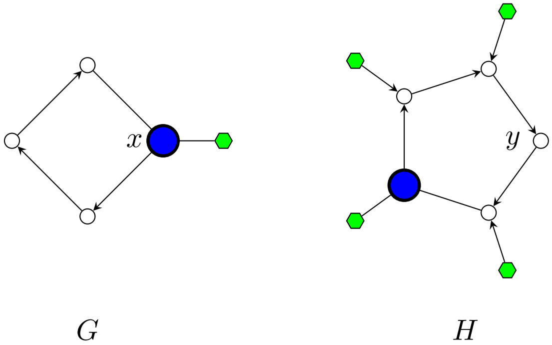

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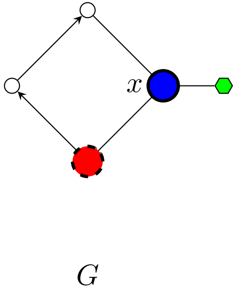

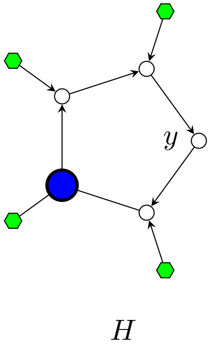

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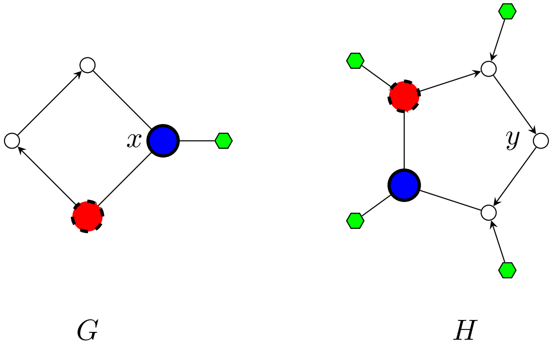
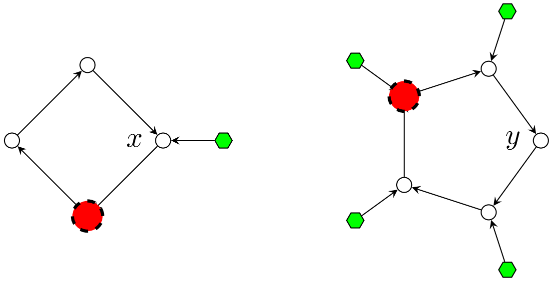


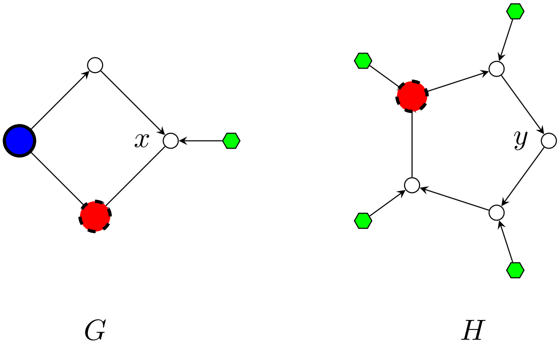
Lifting it to FO^2_a



G

H

Lifting it to FO_a^2



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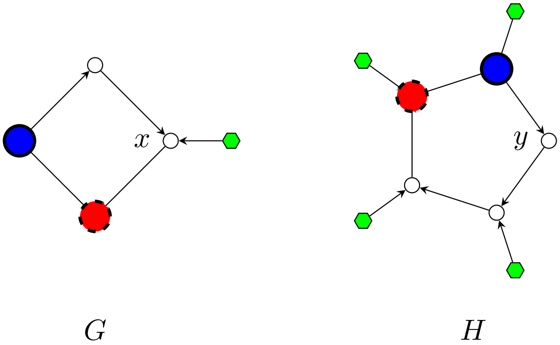

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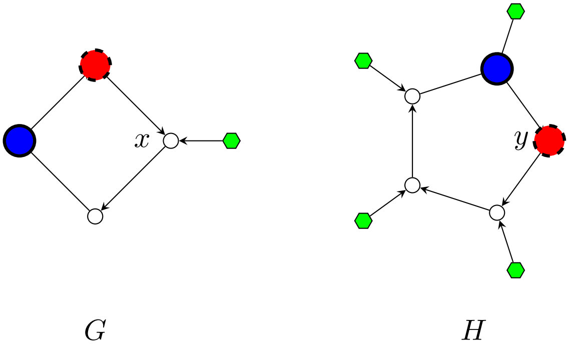
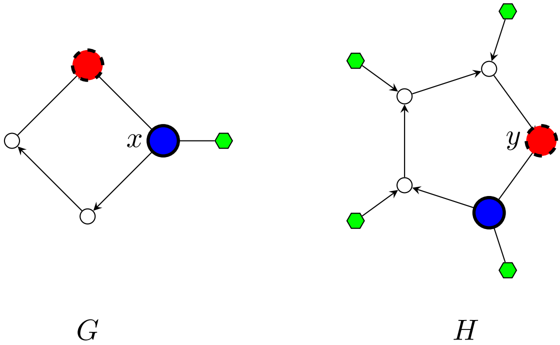
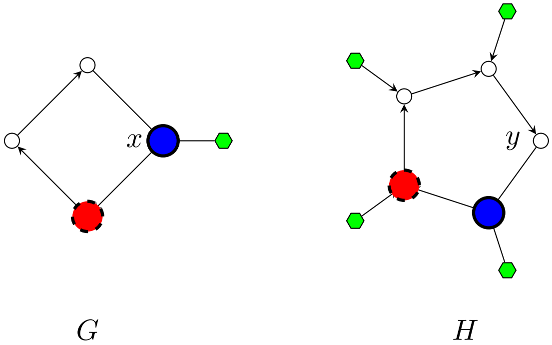



x
y
G
H

Lifting it to FO_a^2



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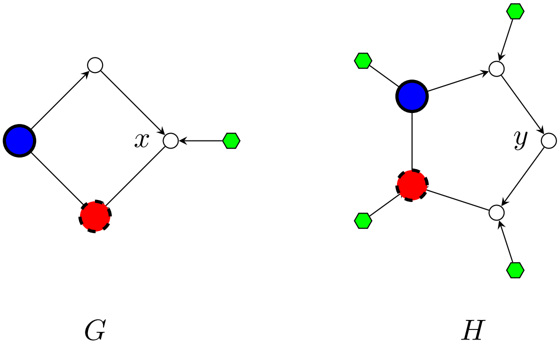

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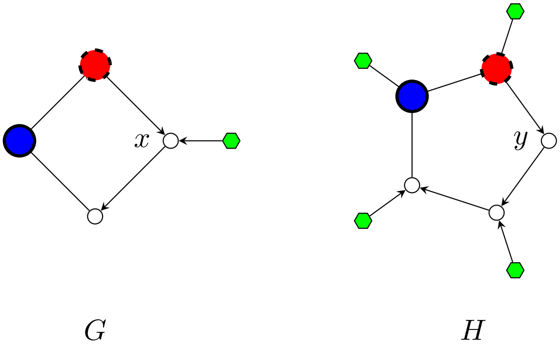
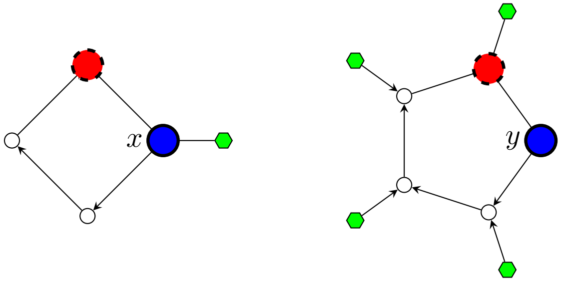
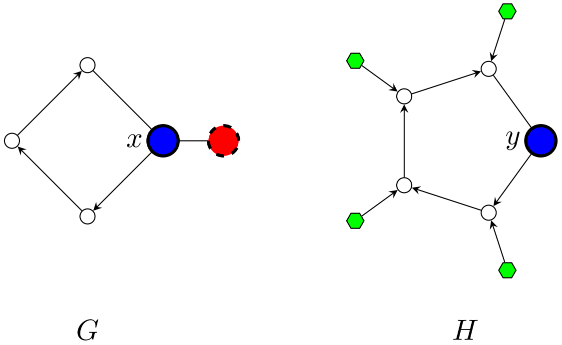


Lifting it to FO^2_a

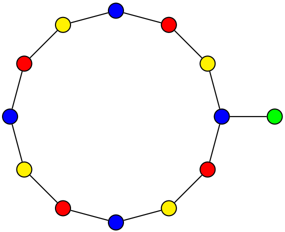


G H

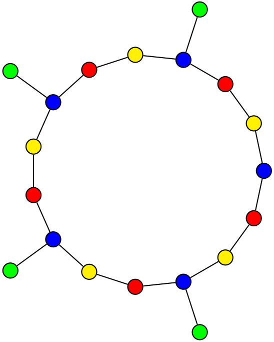

x
y
G
H

Switching to colored graphs

G

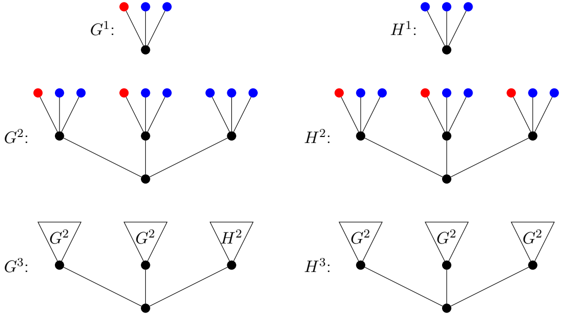


H



Lifting it further to FO_a^2

For FO_a^2, $a > 1$ we apply the tree construction with G and H at the leaves.



Outline

References

- A.K. Chandra and D. Harel. Structure and complexity of relational queries. J. Comput. Syst. Sci., 25:99–128 (1982).
- C. Berkholz, A. Krebs, and O. Verbitsky. Bounds for the quantifier depth in finite-variable logics: Alternation hierarchy. ACM Transactions on Computational Logic 16, Article 21 (2015).