Description logic syntax and semantics

$\mathsf{N_C}$: set of *concept names* (unary predicates, classes) A, B
$\mathsf{N_R}$: set of *role names* (binary predicates, properties) r, s
$\mathsf{N_R^{\pm}} = \{r, r^- \mid r \in \mathsf{N_R}\}$: set of role names and *inverse roles* R, S
$\mathsf{N_I}$: set of individuals names (constants) $a, b, c, \ldots$

Complex concepts (built from $\mathsf{N_C}, \mathsf{N_R}$ using constructors: see below) C, D

TBox (ontology) = set of terminological axioms $\mathcal{T}$
ABox (dataset) = set of ABox assertions ($A(a)$, $r(a,b)$) $\mathcal{A}$
Knowledge base (KB) = TBox + ABox $\mathcal{K}$

Name	**Syntax**	**Semantics**	
Top concept	$\top$	$\Delta^{\mathcal{I}}$	CONCEPTS
Bottom concept	$\bot$	$\emptyset$	
Negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$	
Conjunction	$C_1 \sqcap C_2$	${C_1}^{\mathcal{I}} \cap {C_2}^{\mathcal{I}}$	
Existential restriction	$\exists R.C$	$\{d_1 \mid \text{there exists } (d_1, d_2) \in R^{\mathcal{I}} \text{ with } d_2 \in C^{\mathcal{I}}\}$	
Inverse	r^-	$\{(d_2, d_1) \mid (d_1, d_2) \in r^{\mathcal{I}}\}$	ROLES
Role negation	$\neg R$	$(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \setminus R^{\mathcal{I}}$	
Concept inclusion	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$	TBOX AXIOMS
Role inclusion	$R \sqsubseteq S$	$R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$	
Transitivity axiom	$\mathsf{trans}(R)$	$R^{\mathcal{I}} \cdot R^{\mathcal{I}} \subseteq R^{\mathcal{I}}$	
Concept assertion	$A(a)$	$a^{\mathcal{I}} \in A^{\mathcal{I}}$	ABOX ASSERTIONS
Role assertion	$r(a,b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$	

Horn DLs

DL-Lite$_R$:

- concept inclusions $B_1 \sqsubseteq (\neg)B_2$, with B_1, B_2 of the form $A \in \mathsf{N_C}$ or $\exists R$ ($R \in \mathsf{N_R^{\pm}}$)
- role inclusions $R_1 \sqsubseteq (\neg)R_2$, where $R_1, R_2 \in \mathsf{N_R^{\pm}}$
- note: $\exists R$ can be seen as shorthand for $\exists R.\top$

$\mathcal{EL}$:

- concept constructors: $\top$, $\sqcap$, and $\exists r.C$
- only concept inclusions $C \sqsubseteq D$ in TBox
- normal form: can assume all inclusions of the following forms

$$A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B \quad A \sqsubseteq \exists r.B \quad \exists r.A \sqsubseteq B$$

$\mathcal{ELHI}_\bot$:

- concept constructors: $\top$, $\bot$, $\sqcap$, and $\exists R.C$ ($R \in \mathsf{N}_\mathsf{R}^\pm$)
- both concept inclusions and role inclusions ($R_1 \sqsubseteq R_2$, with $R_1, R_2 \in \mathsf{N}_\mathsf{R}^\pm$)

Certain answers

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a DL KB, and let q be an n-ary query. The set $\mathsf{cert}(q, \mathcal{K})$ of *certain answers* to q over $\mathcal{K}$ is defined as follows:

$$\{(a_1, \ldots, a_n) \in \mathsf{Ind}(\mathcal{A})^n \mid (a_1^\mathcal{I}, \ldots, a_n^\mathcal{I}) \in \mathsf{ans}(q, \mathcal{I}) \text{ for every } \mathcal{I} \in \mathsf{Mods}(\mathcal{K})\}$$

Query rewriting

Let $\mathcal{T}$ be a DL TBox, and let $q, q', q_\bot$ be queries.

- We say that q' is a *rewriting of* q *w.r.t.* $\mathcal{T}$ just in the case that $\mathsf{cert}(q, (\mathcal{T}, \mathcal{A})) = \mathsf{ans}(q', \mathcal{I}_\mathcal{A})$ for every ABox $\mathcal{A}$.
- We call q' a *rewriting of* q *w.r.t.* $\mathcal{T}, \Sigma$ *relative to consistent ABoxes* if $\mathsf{cert}(q, (\mathcal{T}, \mathcal{A})) = \mathsf{ans}(q', \mathcal{I}_\mathcal{A})$ for every ABox $\mathcal{A}$ such that $(\mathcal{T}, \mathcal{A})$ is satisfiable.
- We call $q_\bot$ a *rewriting of unsatisfiability w.r.t.* $\mathcal{T}$ if for every ABox $\mathcal{A}$, we have $\mathsf{cert}(q, (\mathcal{T}, \mathcal{A})) = ()$ iff $(\mathcal{T}, \mathcal{A})$ is unsatisfiable.

Saturation Rules for $\mathcal{EL}$

$$\frac{A \sqsubseteq B_i \ (1 \le i \le n) \quad B_1 \sqcap \ldots \sqcap B_n \sqsubseteq D}{A \sqsubseteq D}\ \textsf{T1} \qquad \frac{A \sqsubseteq B \quad B \sqsubseteq \exists r.D}{A \sqsubseteq \exists r.D}\ \textsf{T2}$$

$$\frac{A \sqsubseteq \exists r.B \quad B \sqsubseteq D \quad \exists r.D \sqsubseteq E}{A \sqsubseteq E}\ \textsf{T3}$$

$$\frac{A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B \quad A_i(a)\ (1 \le i \le n)}{B(a)}\ \textsf{A1} \qquad \frac{\exists r.B \sqsubseteq A \quad r(a,b) \quad B(b)}{A(a)}\ \textsf{A2}$$

Saturation Rules for $\mathcal{ELHI}_\bot$

$$\frac{\{A \sqsubseteq B_i\}_{i=1}^n \quad B_1 \sqcap \ldots \sqcap B_n \sqsubseteq D}{A \sqsubseteq D}\ \textsf{T1} \qquad \frac{R \sqsubseteq S \quad S \sqsubseteq T}{R \sqsubseteq T}\ \textsf{T4} \qquad \frac{M \sqsubseteq \exists R.(N \sqcap \bot)}{M \sqsubseteq \bot}\ \textsf{T5}$$

$$\frac{M \sqsubseteq \exists R.(N \sqcap N') \quad N \sqsubseteq A}{M \sqsubseteq \exists R.(N \sqcap N' \sqcap A)}\ \textsf{T6} \qquad \frac{M \sqsubseteq \exists R.(N \sqcap A) \quad \exists S.A \sqsubseteq B \quad R \sqsubseteq S}{M \sqsubseteq B}\ \textsf{T7}$$

$$\frac{M \sqsubseteq \exists R.N \quad \exists \mathsf{inv}(S).A \sqsubseteq B \quad R \sqsubseteq S}{M \sqcap A \sqsubseteq \exists R.(N \sqcap B)}\ \textsf{T8}$$

$$\frac{A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B \quad A_i(a)\ (1 \le i \le n)}{B(a)}\ \textsf{A1} \qquad \frac{\exists r.B \sqsubseteq A \quad r(a,b) \quad B(b)}{A(a)}\ \textsf{A2}$$

$$\frac{\exists r^-.B \sqsubseteq A \quad r(b,a) \quad B(b)}{A(a)}\ \textsf{A3} \qquad \frac{r \sqsubseteq s \quad r(a,b)}{s(a,b)}\ \textsf{A4} \qquad \frac{r \sqsubseteq s^- \quad r(a,b)}{s(b,a)}\ \textsf{A5}$$