Model counting for logical theories
Problem set 2

1. Prove the *inclusion-exclusion principle*:

$$\mathsf{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<\ldots<i_k\le n} \mathsf{P}(A_{i_1}\ldots A_{i_k}).$$

   Hint: Partition events into disjoint unions of events of the form $A_1^{\sigma_1}\ldots A_n^{\sigma_n}$ where each $A_i^{\sigma_i}$ is either A_i or $\overline{A_i}$. Compare the coefficient at $\mathsf{P}(A_1^{\sigma_1}\ldots A_n^{\sigma_n})$ in the left- and right-hand sides. (Alternatively, you can use induction on n.)

2. Prove the *union bound*: $\mathsf{P}\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{k=1}^{n} \mathsf{P}(A_i)$.
   Hint: We already know that $\mathsf{P}(A\cup B) \le \mathsf{P}(A) + \mathsf{P}(B)$.

3. Let $(\Omega, 2^\Omega, \mathsf{P})$ be a discrete probability space with Ω a finite set. Suppose $B \in 2^\Omega$ is such that $\mathsf{P}(B) > 0$. Prove that the function $\mathsf{Q}\colon 2^\Omega \to \mathbb{R}$ defined by $\mathsf{Q}(A) = \mathsf{P}(A \mid B)$ is a probability measure:
   (a) Show that $\mathsf{Q}(A) \ge 0$ for all $A \in 2^\Omega$.
   (b) Show that $\mathsf{Q}(\Omega) = 1$.
   (c) Show that $\mathsf{Q}(A_1\cup\ldots\cup A_n) = \sum_{i=1}^{n} \mathsf{Q}(A_i)$ if the events $A_1,\ldots,A_n$ are pairwise disjoint, i.e., if $A_i \cap A_j = \varnothing$ for $i \ne j$.

4. From the set of strings $\{000, 001, 002, \ldots, 999\}$ a string $X_1X_2X_3$ is picked uniformly at random. We have proved in class that the events $X_1 = 5$, $X_2 = 5$, and $X_3 = 5$ are independent. What if the original set is replaced with $\{000, 001, 002, \ldots, 998\}$?

5. In class we have constructed a probability space $(\Omega, 2^\Omega, \mathsf{P})$ and three events $A_1, A_2, A_3 \in 2^\Omega$ such that $\mathsf{P}(A_iA_j) = \mathsf{P}(A_i)\,\mathsf{P}(A_j)$ for all $i \ne j$, but A_1, A_2, A_3 are not independent. Can you come up with a (possibly different) probability space and three events B_1, B_2, B_3 such that $\mathsf{P}(B_1B_2B_3) = \mathsf{P}(B_1)\,\mathsf{P}(B_2)\,\mathsf{P}(B_3)$, but B_1, B_2, B_3 are not independent?

6. Can there exist random variables X, Y that have $\mathsf{Cov}(X, Y) = 0$ but are *not* independent? If yes, find an example, otherwise prove this impossible.

7. Verify that the expectation and the variance of a geometrically distributed random variable with parameter $p \in (0, 1)$ are q/p and q/p^2, respectively.