

Probabilistic Program Analysis

Logic and Analysis

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Moore-Penrose Pseudo-Inverse

Definition

Let \mathcal{C} and \mathcal{D} be two Hilbert spaces and $\mathbf{A} : \mathcal{C} \rightarrow \mathcal{D}$ a bounded linear map. A bounded linear map $\mathbf{A}^\dagger = \mathbf{G} : \mathcal{D} \rightarrow \mathcal{C}$ is the **Moore-Penrose pseudo-inverse** of \mathbf{A} iff

- (i) $\mathbf{A} \circ \mathbf{G} = \mathbf{P}_A$,
- (ii) $\mathbf{G} \circ \mathbf{A} = \mathbf{P}_G$,

where \mathbf{P}_A and \mathbf{P}_G denote orthogonal projections onto the ranges of \mathbf{A} and \mathbf{G} .

(Orthogonal) Projections – Idempotents

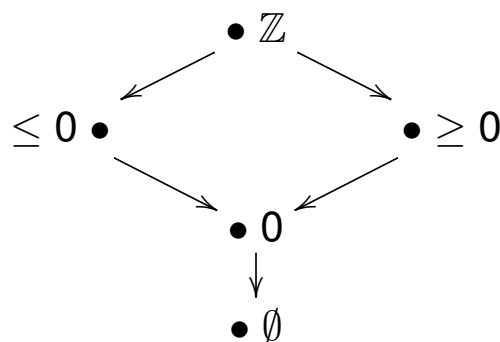
On finite dimensional vector (Hilbert) spaces we have an **inner product** $\langle \cdot, \cdot \rangle$. This allows us to define an **adjoint** via:

$$\langle \mathbf{A}(x), y \rangle = \langle x, \mathbf{A}^*(y) \rangle$$

- An operator \mathbf{A} is **self-adjoint** if $\mathbf{A} = \mathbf{A}^*$.
- An operator \mathbf{A} is **positive**, i.e. $\mathbf{A} \sqsupseteq 0$, if there exists an operator \mathbf{B} such that $\mathbf{A} = \mathbf{B}^* \mathbf{B}$.
- An **(orthogonal) projection** is a self-adjoint \mathbf{E} with $\mathbf{E}\mathbf{E} = \mathbf{E}$.

Projections identify (closed) sub-spaces $Y_{\mathbf{E}} = \{\mathbf{E}x \mid x \in \mathcal{V}\}$.

Example: Sign Domain



Enumeration: $Sign = \{\emptyset, 0, \geq 0, \leq 0, \mathbb{Z}\}$

Free Vector Space: $\mathcal{V}(Sign) = \left\{ \sum_{s \in Sign} x_s \cdot s \mid x_i \in \mathbb{R} \right\}$

Francesca Scozzari: *Domain theory in abstract interpretation: equations, completeness and logic*. PhD Thesis, Siena 1999.

Example: Classical Abstractions (Domains via uco)

Consider the upward closed sub-domains of $\{\emptyset, 0, \geq 0, \leq 0, \mathbb{Z}\}$:

$$\begin{array}{ll}
 \rho_1 = \{\mathbb{Z}\} & \rho_8 = \{\mathbb{Z}, 0, \emptyset\} \\
 \rho_2 = \{\mathbb{Z}, \geq 0\} & \rho_9 = \{\mathbb{Z}, \leq 0, 0\} \\
 \rho_3 = \{\mathbb{Z}, 0\} & \rho_{10} = \{\mathbb{Z}, \leq 0, \emptyset\} \\
 \rho_4 = \{\mathbb{Z}, \emptyset\} & \rho_{11} = \{\mathbb{Z}, \geq 0, 0, \emptyset\} \\
 \rho_5 = \{\mathbb{Z}, \leq 0\} & \rho_{12} = \{\mathbb{Z}, \leq 0, \geq 0, 0, \emptyset\} \\
 \rho_6 = \{\mathbb{Z}, \geq 0, \emptyset\} & \rho_{13} = \{\mathbb{Z}, \leq 0, 0, \emptyset\} \\
 \rho_7 = \{\mathbb{Z}, \geq 0, 0\} & \rho_{14} = \{\mathbb{Z}, \leq 0, \geq 0, 0, \emptyset\}
 \end{array}$$

Identify abstract domains via **upward closed operators** (uco)
 $\rho = \alpha \circ \gamma$ (vs downward closed operators (dco) $\gamma \circ \alpha$).

Example: Probabilistic Abstractions \mathbf{R}_n

$$\begin{array}{ll}
 \mathbf{R}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & \mathbf{R}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
 \mathbf{R}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & \mathbf{R}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
 \mathbf{R}_5 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & \mathbf{R}_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
 \end{array}$$

Example: Probabilistic Abstractions \mathbf{R}_n

$$\mathbf{R}_7 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{R}_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\mathbf{R}_9 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{R}_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: Probabilistic Abstractions \mathbf{R}_n

$$\mathbf{R}_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{R}_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\mathbf{R}_{13} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{R}_{14} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Computing Intersections/Unions

Associate to every PAI (\mathbf{A}, \mathbf{G}) a **projection** (similar to uco):

$$\mathbf{E} = \mathbf{AG} = \mathbf{AA}^\dagger.$$

A general way to construct $\mathbf{E} \sqcap \mathbf{F}$ and (by exploiting de Morgan's law) also $\mathbf{E} \sqcup \mathbf{F} = (\mathbf{E}^\perp \sqcap \mathbf{F}^\perp)^\perp$ is via an infinite approximation sequence and has been suggested by Halmos:

$$\mathbf{E} \sqcap \mathbf{F} = \lim_{n \rightarrow \infty} (\mathbf{EFE})^n.$$

Commutative Case

The concrete construction of $\mathbf{E} \sqcup \mathbf{F}$ and $\mathbf{E} \sqcap \mathbf{F}$ is in general not trivial. Only for **commuting projections** we have:

$$\mathbf{E} \sqcup \mathbf{F} = \mathbf{E} + \mathbf{F} - \mathbf{EF} \text{ and } \mathbf{E} \sqcap \mathbf{F} = \mathbf{EF}.$$

Example

Consider a finite set Ω with a probability structure. For any (measurable) subset A of Ω define the characteristic function χ_A with $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. The characteristic functions are (commutative) projections on random variables using pointwise multiplication, i.e. $X_{\chi_A \chi_A} = X_{\chi_A}$. We have $\chi_{A \cap B} = \chi_A \chi_B$ and $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B$.

Non-Commutative Case

The Moore-Penrose pseudo-inverse is also useful for computing the $\mathbf{E} \sqcap \mathbf{F}$ and $\mathbf{E} \sqcup \mathbf{F}$ of general, non-commuting projections via the **parallel sum**

$$\mathbf{A} : \mathbf{B} = \mathbf{A}(\mathbf{A} + \mathbf{B})^\dagger \mathbf{B}$$

The **intersection of projections** is given by:

$$\mathbf{E} \sqcap \mathbf{F} = 2(\mathbf{E} : \mathbf{F}) = \mathbf{E}(\mathbf{E} + \mathbf{F})^\dagger \mathbf{F} + \mathbf{F}(\mathbf{E} + \mathbf{F})^\dagger \mathbf{E}$$

Israel, Greville: *Generalized Inverses, Theory and Applications*, Springer 03

Projection Operators

Define a **partial order** on self-adjoint operators and projections as follows: $\mathbf{H} \sqsubseteq \mathbf{K}$ iff $\mathbf{K} - \mathbf{H}$ is **positive**, i.e. there exists a \mathbf{B} such that $\mathbf{K} - \mathbf{H} = \mathbf{B}^* \mathbf{B}$.

Alternatively, order projections by inclusion of their image spaces, i.e. $\mathbf{E} \sqsubseteq \mathbf{F}$ iff $Y_{\mathbf{E}} \subseteq Y_{\mathbf{F}}$.

The orthogonal projections form a complete lattice.

The range of the **intersection** $\mathbf{E} \sqcap \mathbf{F}$ is to the closure of the intersection of the image spaces of \mathbf{E} and \mathbf{F} .

The **union** $\mathbf{E} \sqcup \mathbf{F}$ corresponds to the union of the images.

Ortholattices I

Non-distributive analogs of Boolean algebras.

Definition (Ortholattice I)

An *ortholattice* $(L, \sqsubseteq, \cdot^\perp, 0, 1)$ is a lattice (L, \sqsubseteq) with universal bounds 0 and 1, i.e.

- ① (L, \sqsubseteq) is a partial order (i.e. \sqsubseteq is reflexive, antisymmetric, and transitive),
- ② all pairs of elements $a, b \in L$ have a least upper bound (sup) denoted by $a \sqcup b$, and a greatest lower bound (inf) denoted by $a \sqcap b$,
- ③ $0 \sqsubseteq a$ and $a \sqsubseteq 1$ for all $a \in L$.

...

Ortholattices II

Definition (Ortholattice II)

... and a unary *complementation* operation $a \mapsto a^\perp$ satisfying:

- ① $a \sqcap a^\perp = 0$ and $a \sqcup a^\perp = 1$ for all $a \in L$,
- ② $(a \sqcap b)^\perp = a^\perp \sqcup b^\perp$ and $(a \sqcup b)^\perp = a^\perp \sqcap b^\perp$ for all $a, b \in L$,
- ③ $(a^\perp)^\perp = a$ for all $a \in L$.

The set $P(\mathcal{H})$ of closed-range projections on a Hilbert space \mathcal{H} is a non-distributive ortholattice

$$\langle P(\mathcal{H}), \sqsubseteq, \sqcup, \sqcap, \cdot^\perp, \mathbf{1}, \mathbf{0} \rangle$$

Commutativity and Distributivity

In general, \cap and \sqcup in an ortholattice are **not distributive**, ie.

$$(a \cap b) \sqcup (a \cap c) \not\subseteq a \cap (b \sqcup c)$$

$$a \sqcup (b \cap c) \not\subseteq (a \sqcup b) \cap (a \sqcup c)$$

Two elements a and b in an ortholattice **commute**, denoted by $[a, b] = 0$, iff

$$a = (a \cap b) \sqcup (a \cap b^\perp)$$

An ortholattice is called an **orthomodular lattice** if $[a, b] = 0$ implies $[b, a] = 0$.

Example: Projections $\mathbf{P}_n = \mathbf{R}_n \mathbf{R}_n^\dagger$

$$\mathbf{P}_1 = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$\mathbf{P}_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \mathbf{P}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$\mathbf{P}_5 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \mathbf{P}_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Example: Projections $\mathbf{P}_n = \mathbf{R}_n \mathbf{R}_n^\dagger$

$$\mathbf{P}_7 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{P}_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

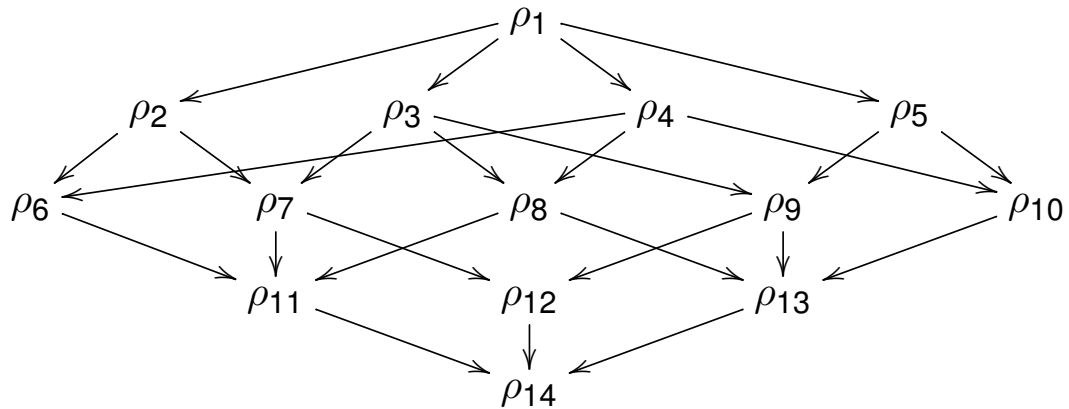
$$\mathbf{P}_9 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{P}_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: Projections $\mathbf{P}_n = \mathbf{R}_n \mathbf{R}_n^\dagger$

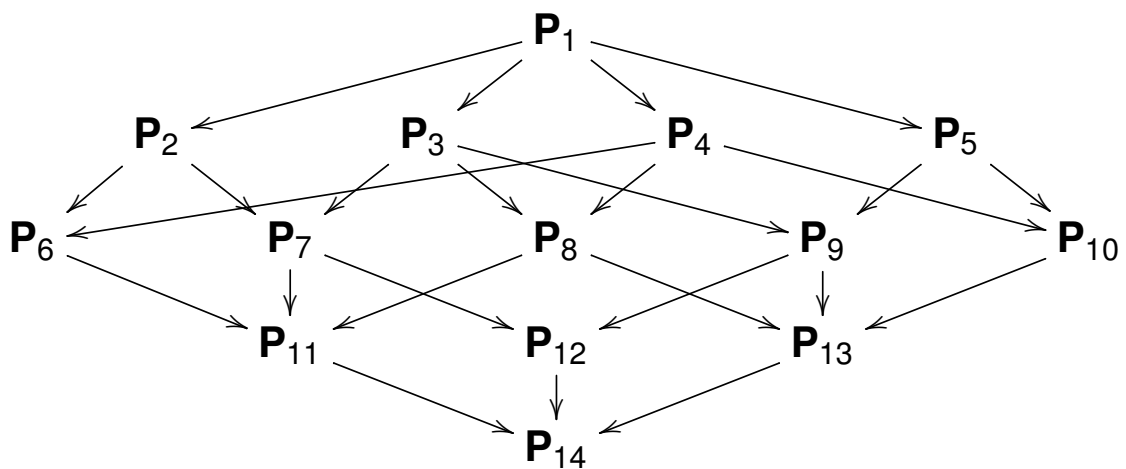
$$\mathbf{P}_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{P}_{12} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{P}_{13} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{P}_{14} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: The Lattice $uco(\text{Sign})$



Example: The Lattice $\mathcal{P}(\mathcal{V}(\text{Sign}))$



Example: Combining Projections

$$\begin{aligned}
 \mathbf{P}_7 \sqcap \mathbf{P}_8 &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \sqcap \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \\
 &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \mathbf{P}_3
 \end{aligned}$$

In particular, we have $\mathbf{P}_7 \sqcap \mathbf{P}_8 = \mathbf{P}_7 \mathbf{P}_8$ as \mathbf{P}_7 and \mathbf{P}_8 commute, i.e. $[\mathbf{P}_7, \mathbf{P}_8] = \mathbf{P}_7 \mathbf{P}_8 - \mathbf{P}_8 \mathbf{P}_7 = \mathbf{O}$.

Example: Combining Projections

$$\begin{aligned}
 \mathbf{P}_4 \sqcap \mathbf{P}_7 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \sqcap \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \\
 &= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} = \mathbf{P}_1
 \end{aligned}$$

Using the expression $\mathbf{P}_4 \sqcap \mathbf{P}_7 = 2\mathbf{P}_4(\mathbf{P}_4 + \mathbf{P}_7)^\dagger \mathbf{P}_7$ as \mathbf{P}_4 and \mathbf{P}_7 do not commute.

Example: Combining Projections

Note that the simple multiplication $\mathbf{P}_4\mathbf{P}_7$ is different from $\mathbf{P}_4 \sqcap \mathbf{P}_7$:

$$\begin{aligned}\mathbf{P}_4\mathbf{P}_7 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \\ &= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \neq \mathbf{P}_4 \sqcap \mathbf{P}_7\end{aligned}$$

Precision Measures

Definition

Given two vector (Hilbert) spaces \mathcal{C} and \mathcal{D} and a bounded linear map $\mathbf{F} : \mathcal{C} \rightarrow \mathcal{D}$, then we say that a pair of projections $\mathbf{P} : \mathcal{C} \rightarrow \mathcal{C}$ and $\mathbf{R} : \mathcal{D} \rightarrow \mathcal{D}$ is **complete** for \mathbf{F} iff

$$\mathbf{FP} = \mathbf{RFP}.$$

Given a pair of projections (\mathbf{P}, \mathbf{R}) for a function \mathbf{F} , we estimate the **precision** of the abstraction via the “difference” between \mathbf{FP} and its optimal version \mathbf{RFP} .

$$\text{Prec}_{\mathbf{F}}(\mathbf{P}, \mathbf{R}) = \|\mathbf{FP} - \mathbf{RFP}\|.$$

Order and Precision

Proposition

Let $\mathbf{F} : \mathcal{H}_1 \mapsto \mathcal{H}_2$ be a bounded linear operator between two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , and let $\mathbf{P}_1, \mathbf{P}_2 \in P(\mathcal{H}_2)$ and $\mathbf{R} \in P(\mathcal{H}_1)$.

Then we have: if $\mathbf{P}_1 \sqsubseteq \mathbf{P}_2$ then $\text{Prec}_{\mathbf{F}}(\mathbf{P}_1, \mathbf{R}) \leq \text{Prec}_{\mathbf{F}}(\mathbf{P}_2, \mathbf{R})$.

Example: (Relative) Precisions

	\mathbf{P}_1	\mathbf{P}_2	\mathbf{P}_3	\mathbf{P}_4	\mathbf{P}_5	\mathbf{P}_6	\mathbf{P}_7	\mathbf{P}_8	\mathbf{P}_9	\mathbf{P}_{10}	\mathbf{P}_{11}	\mathbf{P}_{12}	\mathbf{P}_{13}	\mathbf{P}_{14}
\mathbf{P}_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
\mathbf{P}_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
\mathbf{P}_3	1	.75	0	.79	.75	.65	0	0	0	.65	0	0	0	0
\mathbf{P}_4	1	.91	.79	0	.91	0	.79	0	.79	0	0	.79	0	0
\mathbf{P}_5	1	.75	0	.79	.75	.65	0	0	0	.65	0	0	0	0
\mathbf{P}_6	1.10	1	.87	0	1	0	.87	0	.87	0	0	.87	0	0
\mathbf{P}_7	1.34	1	0	1.06	1	.87	0	0	0	.87	0	0	0	0
\mathbf{P}_8	1	1	1	1	1	.82	1	0	1	.82	0	1	0	0
\mathbf{P}_9	1.10	.82	0	.87	.82	.71	0	0	0	.71	0	0	0	0
\mathbf{P}_{10}	1.07	.91	.87	.87	.91	.71	.87	0	.87	.71	0	.87	0	0
\mathbf{P}_{11}	1.34	1	1	1.22	1	1	1	0	1	1	0	1	0	0
\mathbf{P}_{12}	1.34	1	0	1.06	1	.87	0	0	0	.87	0	0	0	0
\mathbf{P}_{13}	1.10	1	1	1.06	1	.87	1	0	1	.87	0	1	0	0
\mathbf{P}_{14}	1.34	1	1	1.22	1	1	1	0	1	1	0	1	0	0

Linear Operator Semantics (LOS)

The **collecting semantics** of a program P is given by:

$$\mathbf{T}(P) = \sum p_{ij} \cdot \mathbf{T}(\ell_i, \ell_j)$$

Local effects $\mathbf{T}(\ell_i, \ell_j)$: Data Update + Control Step

$$\mathbf{T}(\ell_i, \ell_j) = (\mathbf{N}_{i1} \otimes \mathbf{N}_{i2} \otimes \dots \otimes \mathbf{N}_{iv}) \otimes \mathbf{M}_{ij}$$

Kronecker Products

Given a $n \times m$ matrix \mathbf{A} and a $k \times l$ matrix \mathbf{B} :

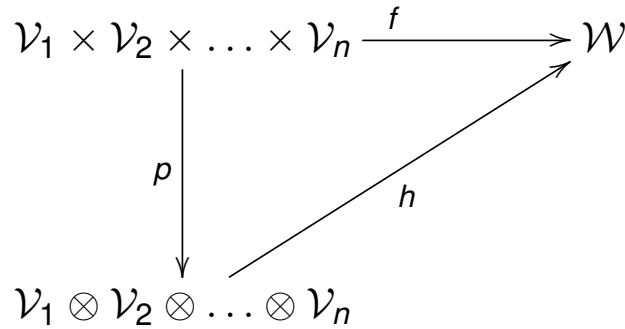
$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1m} & \dots & a_{nm} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b_{11} & \dots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{1l} & \dots & b_{kl} \end{pmatrix}$$

The **tensor product** $\mathbf{A} \otimes \mathbf{B}$ is then a $nk \times ml$ matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{1m}\mathbf{B} & \dots & a_{nm}\mathbf{B} \end{pmatrix}$$

Abstract Tensor Product

The (algebraic) **tensor product** of vector spaces $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_n$ is given by a vector space $\bigotimes_{i=1}^n \mathcal{V}_i$ and a map $p = \bigotimes_{i=1}^n \rho_i \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_n; \bigotimes_{i=1}^n \mathcal{V}_i)$ such that if \mathcal{W} is any vector space and $f \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_n; \mathcal{W})$ then there exists a unique map $h : \bigotimes_{i=1}^n \mathcal{V}_i \rightarrow \mathcal{W}$ satisfying $f = h \circ p$.



$$\mathcal{V}(X \times Y) = \mathcal{V}(X) \otimes \mathcal{V}(Y)$$

Tensor Product Properties

The tensor product of n linear operators $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ is associative (but in general not commutative) and has e.g. the following properties:

- 1 $(\mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_n) \cdot (\mathbf{B}_1 \otimes \dots \otimes \mathbf{B}_n) = \mathbf{A}_1 \cdot \mathbf{B}_1 \otimes \dots \otimes \mathbf{A}_n \cdot \mathbf{B}_n$
- 2 $\mathbf{A}_1 \otimes \dots \otimes (\alpha \mathbf{A}_i) \otimes \dots \otimes \mathbf{A}_n = \alpha (\mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_i \otimes \dots \otimes \mathbf{A}_n)$
- 3 $\mathbf{A}_1 \otimes \dots \otimes (\mathbf{A}_i + \mathbf{B}_i) \otimes \dots \otimes \mathbf{A}_n = (\mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_i \otimes \dots \otimes \mathbf{A}_n) + (\mathbf{A}_1 \otimes \dots \otimes \mathbf{B}_i \otimes \dots \otimes \mathbf{A}_n)$
- 4 $(\mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_i \otimes \dots \otimes \mathbf{A}_n)^\dagger = \mathbf{A}_1^\dagger \otimes \dots \otimes \mathbf{A}_i^\dagger \otimes \dots \otimes \mathbf{A}_n^\dagger$

Relational Dependency

```
1:  $[m \leftarrow 1]^1$ ;  
2: while  $[n > 1]^2$  do  
3:    $[m \leftarrow m \times n]^3$ ;  
4:    $[n \leftarrow n - 1]^4$   
5: end while  
6: [stop]5
```

Input/output behaviour: Parity of m for different values of n .

- Probability that $m = \text{even/odd}$ **and** $n = 1, 2, 3$.
 - Probability that m is even/odd, and
 - Probability that n is 1, 2, 3.
- Probability that m is even/odd **for** $n = 1, 2, 3$.

Dependency and Correlations

Some **joint probability distributions** can be expressed as tensor product of two (independent) probability distributions \mathbf{e} and \mathbf{f} :

$$\begin{pmatrix} \frac{2}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \otimes \left(\frac{2}{3}, \frac{1}{3}\right)^t$$

But there are **no** two vectors \mathbf{e} and \mathbf{f} such that for example

$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \end{pmatrix} = \mathbf{e} \otimes \mathbf{f}$$

Fully, Weakly and Non-Relational Analysis

Consider compositional (probabilistic) **abstractions** of the form:

$$\mathbf{S} = \bigoplus_{i=1}^v \mathbf{S}(x_i) \quad \text{with} \quad \mathbf{S}(x_i) = \left(\bigotimes_{k=1}^{i-1} \mathbf{S}_{\neg i} \right) \otimes \mathbf{S}_i \otimes \left(\bigotimes_{k=i+1}^v \mathbf{S}_{\neg i} \right)$$

Fully Relational: \mathbf{S}_r is \mathbf{S} with $\mathbf{S}_i = \mathbf{A}_i$ and $\mathbf{S}_{\neg i} = \mathbf{A}_{\neg i}$

Weakly Relational: \mathbf{S}_w is \mathbf{S} with $\mathbf{S}_i = \mathbf{A}_i$ and $\mathbf{S}_{\neg i} = \mathbf{A}_{\neg i}$ or \mathbf{A}_f

Non-Relational: \mathbf{S}_n is \mathbf{S} with $\mathbf{S}_i = \mathbf{A}$ and $\mathbf{S}_{\neg i} = \mathbf{A}_f$

With \mathbf{A}_f forgetful and \mathbf{A}_i and $\mathbf{A}_{\neg i}$ nontrivial abstractions.

For \mathbf{S}_r all factors in \bigoplus are the same; we can take $\mathbf{S}_r = \mathbf{S}(x_1)$.

Examples

```
var x:[0..10]; begin x:=k; stop (k = 1,4)
```

$\mathbf{P} \setminus \mathbf{R}$	\emptyset	\mathbf{S}_n	\mathbf{S}_w	\mathbf{S}_r	id
\emptyset	0	0	0	0	0
\mathbf{S}_n	1.58	0	0	0	0
\mathbf{S}_w	1.58	0	0	0	0
\mathbf{S}_r	1.58	0	0	0	0
id	2.55	1	1	1	0

```
var x:[0..10]; y:[0..10]; begin x:=y; stop
```

$\mathbf{P} \setminus \mathbf{R}$	\emptyset	\mathbf{S}_n	\mathbf{S}_w	\mathbf{S}_r	id
\emptyset	0	0	0	0	0
\mathbf{S}_n	1.73	0	0	0	0
\mathbf{S}_w	2.24	1	0	0	0
\mathbf{S}_r	2.24	1	1	0	0
id	3.61	3.61	3.61	3.61	0

Examples (ctd.)

```
var x:[0..10]; y:[0..3]; begin x:=2*y; stop
```

P\R	\emptyset	S_n	S_w	S_r	id
\emptyset	0	0	0	0	0
S_n	1.88	0.89	0.89	0.89	0
S_w	2.14	1.52	1.29	1.29	0
S_r	2.24	1.64	1.50	1.41	0
id	3.61	3.60	3.59	3.58	0

```
var x:[0..10]; y:[0..3]; begin x:=3*y; stop
```

P\R	\emptyset	S_n	S_w	S_r	id
\emptyset	0	0	0	0	0
S_n	1.77	0.89	0.89	0.89	0
S_w	2.24	1.52	1.29	1.29	0
S_r	2.24	1.64	1.50	1.41	0
id	3.61	3.60	3.59	3.58	0

Examples (ctd.)

Using **cast** d abstraction : \mathbf{A}_d lifted $\alpha(x) = x \bmod d$

S_n is **S** with **S_i** = **S₄**, **S_{-i}** = **A₁**

S_w is **S** with **S_i** = **S₄**, **S_{-i}** = **A₂**

S_r is **S** with **S_i** = **S_{-i}** = **A₄**

Further Work

Conclusions

Conclusions

Some applications of PAI:

- **Approximate Process Equivalences**: The semantics of concurrent processes can be defined via **approximate** equivalences (e.g. ϵ -bisimulation).
- **Approximate Confinement**: Static analysis of **security properties** can be sometimes more effective if the security is guaranteed only up to some acceptable percentage threshold.
- **Probabilistic Program Transformation**: Transforming out **timing leaks**... probabilistically.
- ...

LOS for Variable Probabilities

In every choice construct one must make a choice and the probabilities of all choices must sum up to one (certainty). One can't assume (that the programmer used) normalised probabilities.

We therefore need to normalise probabilities with respect to a context of "competing" probabilities:

$$\tilde{p} = P_{[p_1 \dots p_n]} = \frac{p}{p_1 + \dots + p_n}.$$

This can be done at compile-time if all probabilities are constants, but also at runtime in the operational semantics.

Typically one would assume $p_i \in \mathbb{R}$ or $p_i \in \mathbb{Q}$. However, we can also use **discrete probabilities**, i.e. $p_i \in \mathbb{Z}$.

Duel at High Noon

Consider a "duel" between two cowboys:

- Cowboy A – hitting probability a
- Cowboy B – hitting probability b

- 1 Choose (non-deterministically) whether A or B starts.
- 2 Repeat until winner is known:
 - If it is A 's turn he will hit/shoot B with probability a ;
If B is shot then A is the winner, otherwise it's B 's turn.
 - If it is B 's turn he will hit/shoot A with probability b ;
If A is shot then B is the winner, otherwise it's A 's turn.

Question: What is the life expectancy of A or B ?

Question: What happens if A is learning to shoot better during the duel? How can we model **dynamic probabilities**?

Introduced by McIver and Morgan (2005).

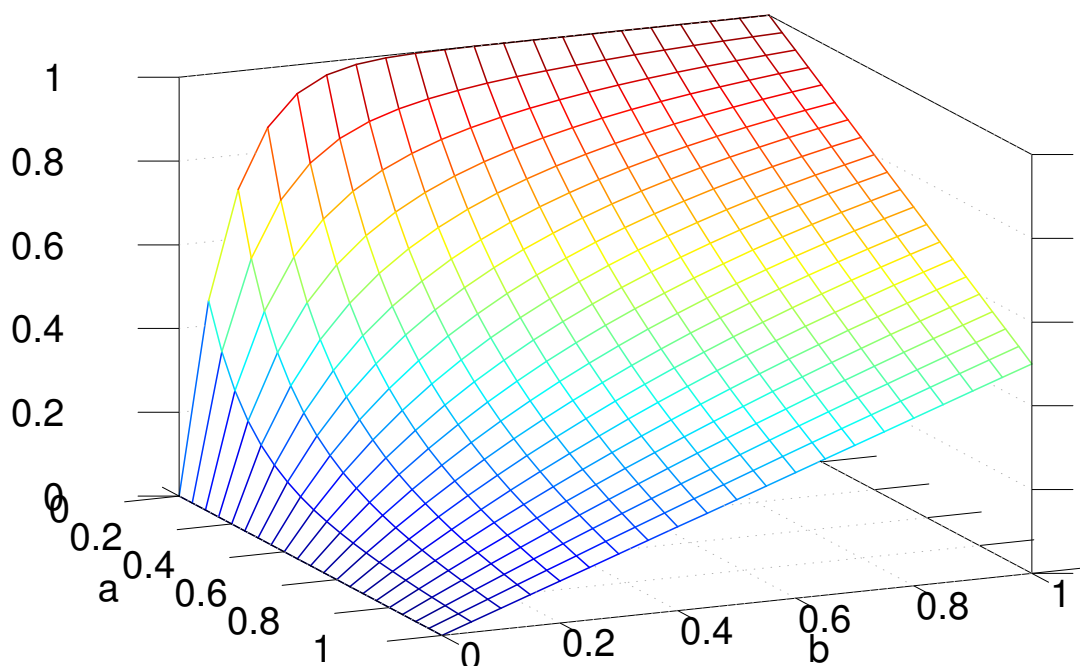
Discussed in detail by Gretz, Katoen, McIver (2012)

Example: Duelling Cowboys

```
begin
  # who's first turn
  choose 1:{t:=0} or 1:{t:=1} ro;
  # continue until ...
  c := 1;
  while c == 1 do
    if (t==0) then
      choose ak:{c:=0} or am:{t:=1} ro
    else
      choose bk:{c:=0} or bm:{t:=0} ro
    fi;
  od;
  stop; # terminal loop
end
```

Example: Duelling Cowboys

The survival chances, i.e. winning probability, for A.



Contexts: Advance Normalisation

For all possible values of the variable probabilities p_i compute their normalisation, compute the possible **contexts**.

$$C[p_1, \dots, p_n] = \begin{cases} \emptyset & \text{if } n = 0 \\ \{[p_1]\} & \text{if } n = 1 \text{ and } p_i \text{ const} \\ \{[c] \mid c \in \mathbf{Value}(p_1)\} & \text{if } n = 1 \text{ and } p_i \text{ var} \\ \bigcup_{[i] \in C[p_1]} \{[i] \cdot C[p_2, \dots, p_n]\} & \text{otherwise, i.e. } n > 1. \end{cases}$$

Example

Variable x with $\mathbf{Value}(x) = \{0, 1\}$ and a parameter $p = 0$ or $p = 1$ then contexts are given by:

$$C[x, 1, p] = \{[0, 1, 0], [1, 1, 0]\} \text{ and } C[x, 0, p] = \{[0, 1, 1], [1, 1, 1]\}$$

Dynamic Probabilities

For all possible values of the variable probabilities test if the current state. With $c_j \in \mathbf{Value}(p_j)$ and $d_j \in \mathbf{Value}(p_j)$ use:

$$P_{c_j[d_1 \dots d_n]}^{p_j[p_1 \dots p_n]} = \mathbf{P}(p_i = c_j) \cdot \left(\prod_{k=1, \dots, n} \mathbf{P}(p_k = d_k) \right)$$

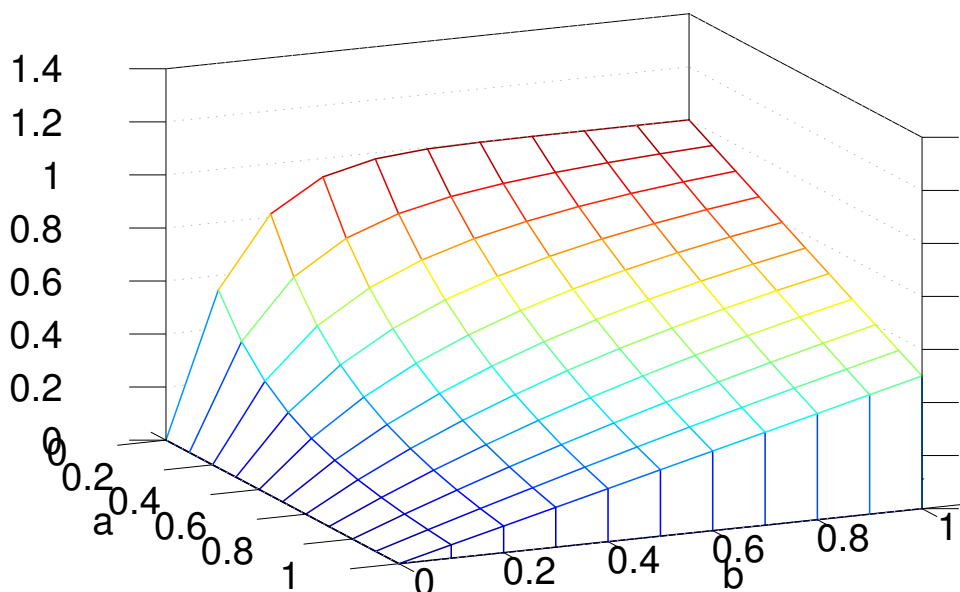
This gives the LOS Semantics for variable probabilities:

$$\{[\mathbf{choose}]^{p_1:S_1} \dots \text{or } p_n : S_n \text{ or } \ell\}_{LOS} = \{S_i\}_{LOS} \cup \bigcup_{i=1}^n \left\{ \sum_{c_j \in \mathbf{Value}(p_j)} \sum_{[d_1 \dots d_n] \in C[p_1 \dots p_n]} c_j[d_1 \dots d_n] \cdot P_{c_j[d_1 \dots d_n]}^{p_j[p_1 \dots p_n]} \otimes \mathbf{E}(\ell, \text{init}(S_i)) \right\}$$

Learning how to shoot straight

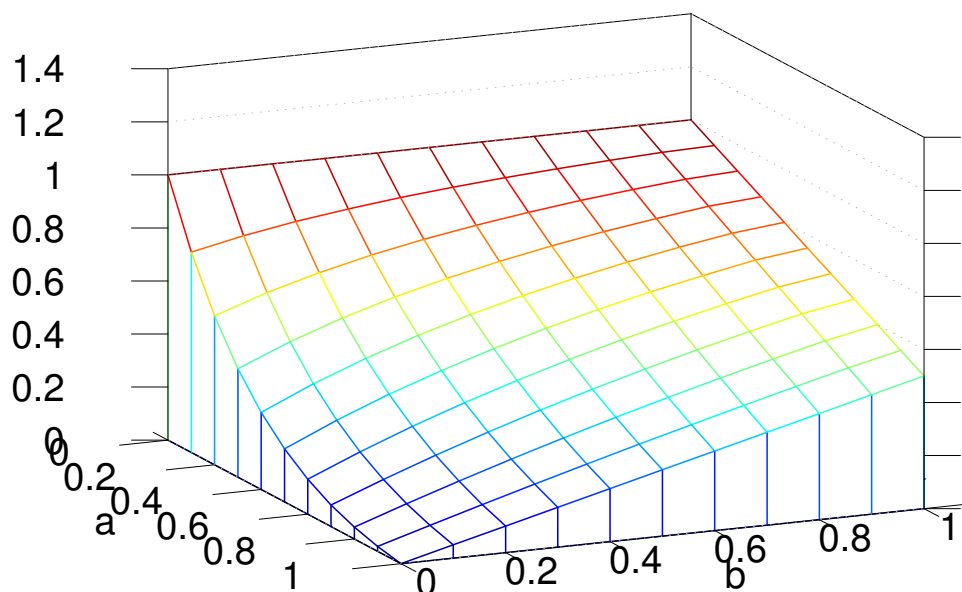
```
begin
# initialise skills of A
akl := ak; aml := am;
# who's first
choose 1:{t:=0} or 1:{t:=1} ro;
# continue until ...
c := 1;
while c == 1 do
  if (t==0) then
    choose akl:{c:=0} or aml:{t:=1} ro
  else
    choose bk:{c:=0} or bm:{t:=0} ro
  fi;
  akl := @inc(akl); aml := @dec(aml);
od;
stop; # terminal loop
end
```

Back to the two Cowboys



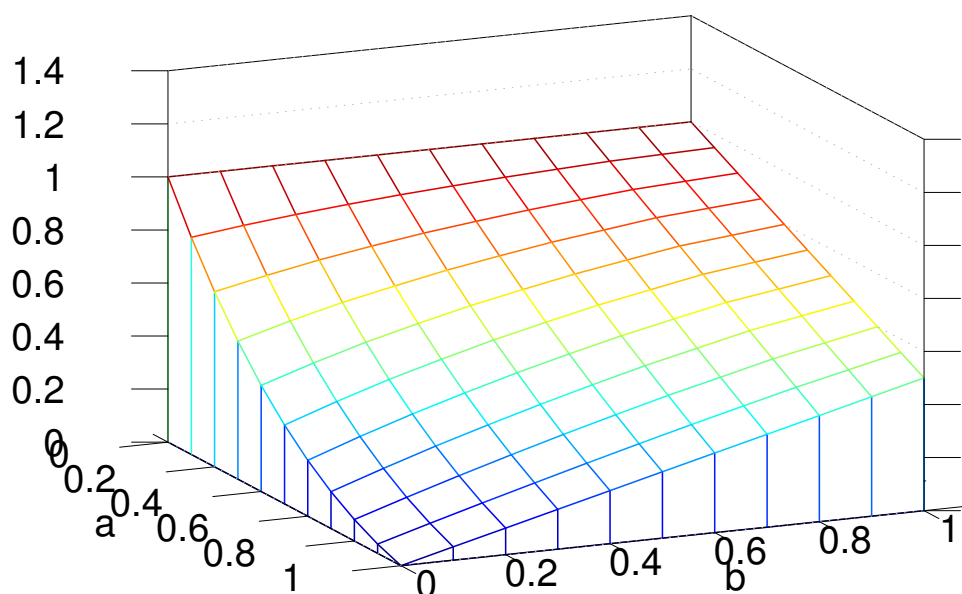
Learning rate **0**.

Back to the two Cowboys



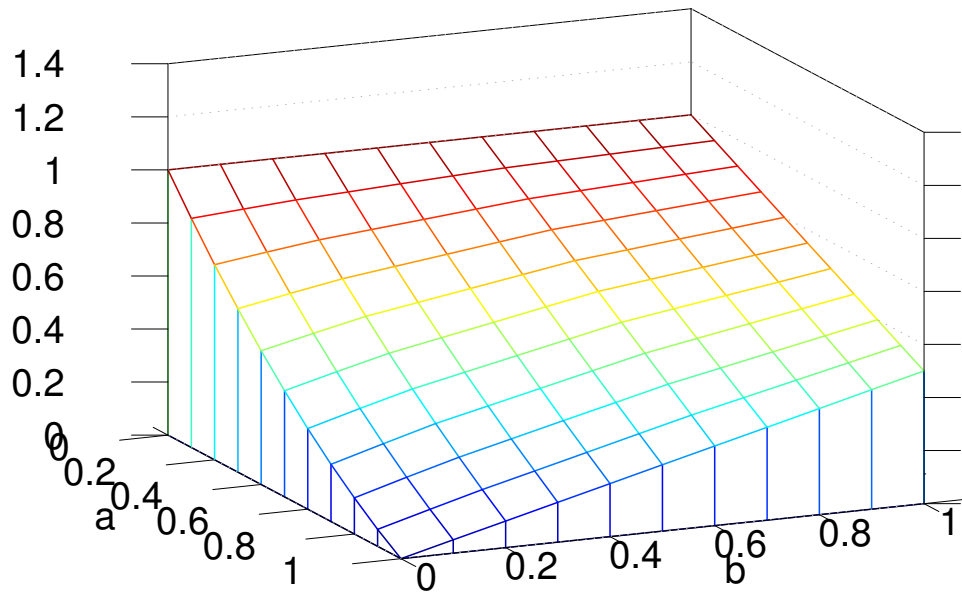
Learning rate 1.

Back to the two Cowboys



Learning rate 2.

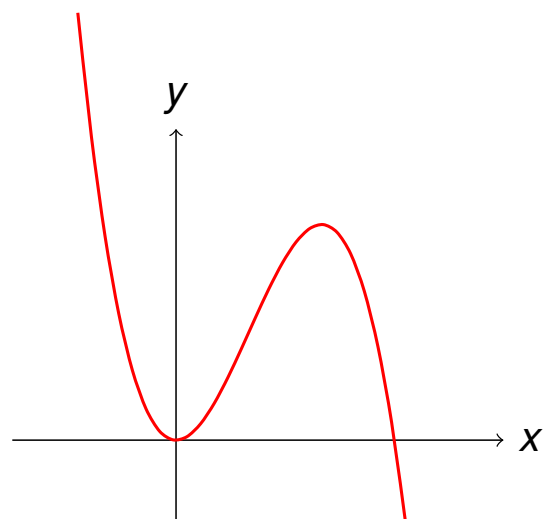
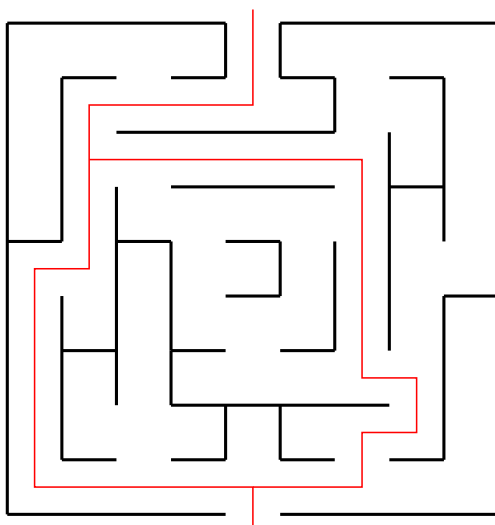
Back to the two Cowboys



Learning rate 4.

LOS for Program Synthesis

Finding the minimum length path vs minimum value of functions



As usual (for now): Take the best non-linear optimisation tool money can't buy (leave it to "them" to make it work).

A General Approach

- Consider **parameterised** program $P(p_1, p_2, \dots, p_n)$ with

... [choose]^ℓ $p_1 : S_1$ or ... or $p_n : S_n$ ro [choose]^ℓ $\lambda_1 : S_1$ or ... or $\lambda_n : S_n$

- Construct the **parametric** LOS semantics/operator, i.e.

$$\llbracket P(\lambda_1, \lambda_2, \dots, \lambda_n) \rrbracket = \mathbf{T}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

- Establish constraints on **functional** behaviour, e.g.

$$\mathbf{A}^\dagger \mathbf{T}(\lambda_1, \lambda_2, \dots, \lambda_n) \mathbf{A} = \llbracket S \rrbracket$$

- Additional **non-functional** (performance) objectives

$$\min_{\lambda_1, \lambda_2, \dots, \lambda_n} \Phi(\mathbf{T}(\lambda_1, \lambda_2, \dots, \lambda_n))$$

Swapping: The XOR Trick

Consider the (probabilistic) **sketch** for swapping x and y :

$$\begin{aligned} & [\text{choose}]^1 \lambda_{1,1} : S_1 \text{ or } \dots \text{ or } \lambda_{1,n} : S_n \text{ ro;} \\ & [\text{choose}]^2 \lambda_{2,1} : S_1 \text{ or } \dots \text{ or } \lambda_{2,n} : S_n \text{ ro;} \\ & [\text{choose}]^3 \lambda_{3,1} : S_1 \text{ or } \dots \text{ or } \lambda_{3,n} : S_n \text{ ro;} \end{aligned}$$

with S_i one of $i = 1, \dots, 13$ different elementary blocks:

$$\begin{aligned} & [\text{skip}]^1 \\ & [x := y]^2 [x := z]^3 \\ & [y := x]^4 [y := z]^5 \\ & [z := x]^6 [z := y]^7 \\ & [x := (x + y) \bmod 2]^8 [x := (x + z) \bmod 2]^9 \\ & [y := (y + x) \bmod 2]^{10} [y := (y + z) \bmod 2]^{11} \\ & [z := (z + x) \bmod 2]^{12} [z := (z + y) \bmod 2]^{13} \end{aligned}$$

Swapping: Parameterised LOS and Objective

Using 13 transfer functions $\mathbf{F}_1 \dots \mathbf{F}_{13}$ to define

$$\mathbf{T}(\lambda_{ij}) = \prod_{i=1}^3 \mathbf{T}_i(\lambda_{ij}) \quad \text{with} \quad \mathbf{T}_i(\lambda_{ij}) = \sum_{j=1}^{13} \lambda_{ij} \mathbf{F}_j$$

For one-bit variables x, y the intended behaviour (on $\mathbb{R}^2 \otimes \mathbb{R}^2$):

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{array}{ll} x \mapsto 0 & y \mapsto 0 \\ x \mapsto 0 & y \mapsto 1 \\ x \mapsto 1 & y \mapsto 0 \\ x \mapsto 1 & y \mapsto 1 \end{array}$$

Objective: $\min \Phi_{00}(\lambda_{ij}) = \|\mathbf{A}^\dagger \mathbf{T}(\lambda_{ij}) \mathbf{A} - \mathbf{S}\|_2$ or $\min \Phi_{\rho\omega}(\lambda_{ij})$ which also penalises for reading or writing to z ; using the abstraction $\mathbf{A} = \mathbf{I}_{(4)} \otimes \mathbf{A}_{f(2)} = \text{diag}(1, 1, 1, 1) \otimes (1, 1)^t$.

Swapping: Test Runs

Using `octave`: if we start with a swap which uses z , like

$$[z := x]^6; [x := y]^2; [y := z]^5$$

represented by λ_{ij} given as:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $\min \Phi_{00}$ we get no change; but with $\min \Phi_{11}$ (after 12 iterations) we get with `octave` the optimal λ_{ij} 's:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

This corresponds to the program:

$$[y := (y+x) \bmod 2]^{10}; [x := (x+y) \bmod 2]^8; [y := (y+x) \bmod 2]^{10}$$

Swapping: Test Runs

For randomly chosen initial values for λ_{ij} :

$$\begin{pmatrix} .70 & .30 & .72 & .84 & .51 & .70 & .76 & .47 & .63 & .63 & .93 & .55 & .68 \\ .74 & .22 & .37 & .70 & .67 & .13 & .93 & .69 & .30 & .88 & .03 & .52 & .80 \\ .59 & .49 & .01 & .69 & .22 & .23 & .10 & .01 & .10 & .22 & .03 & .55 & .11 \end{pmatrix}$$

For $\min \Phi_{11}$ (after 9 iterations) we get the optimal λ_{ij} 's:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

This corresponds to the program:

$[y := (y+x) \bmod 2]^{10}; [x := (x+y) \bmod 2]^8; [y := (y+x) \bmod 2]^{10}$

For Φ_{00} we may also get: $[z := x]^6; [x := y]^2; [y := z]^5$.

Some References

- Di Pierro, Wiklicky: A logico-algebraic approach to probabilistic program analysis Pre-Proceedings of LOPSTR'05.
- Di Pierro, Sotin, Wiklicky: *Relational analysis and precision via probabilistic abstract interpretation*. In QAPL'08 – Workshop on Quantitative Aspects of Programming Languages, ENTCS Elsevier, 2008.
- Wiklicky: *On Dynamical Probabilities, or: How to Learn to Shoot Straight*. in Proceedings of Coordination'16, LNCS 9686, Springer, 2016.
- Israel and Greville: *Generalized Inverses – Theory and Applications*. CMS Books in Mathematics, Springer, 2003.