day 1

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Logical foundations of databases

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About the speakers…

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Works in LaBRI, Bordeaux
CNRS researcher

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post-docs in Warsaw, Edinburgh
Works in LaBRI, Bordeaux
CNRS researcher

First and foremost…

interrupt!

Organization

Schedule:

Relational databases

Relational Algebra

First-Order logic

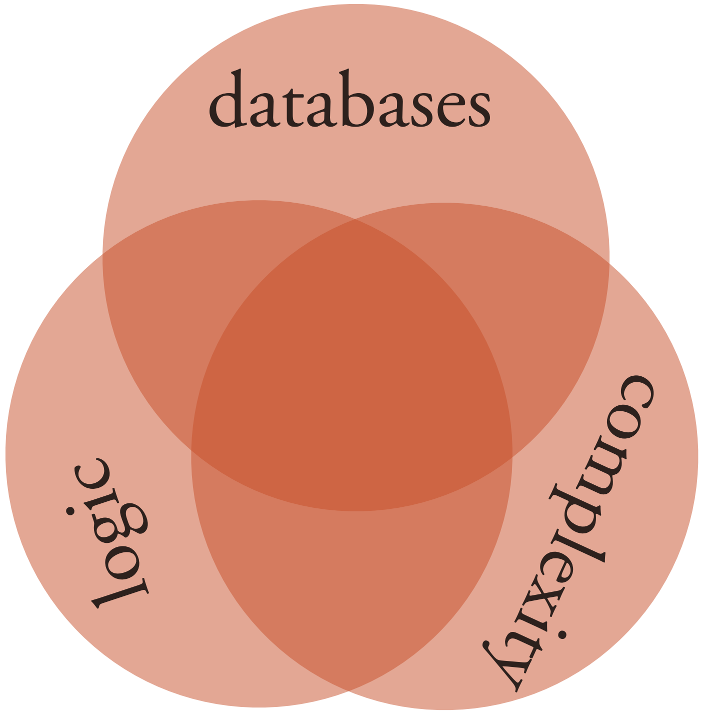
EF games

Locality

0-1 law

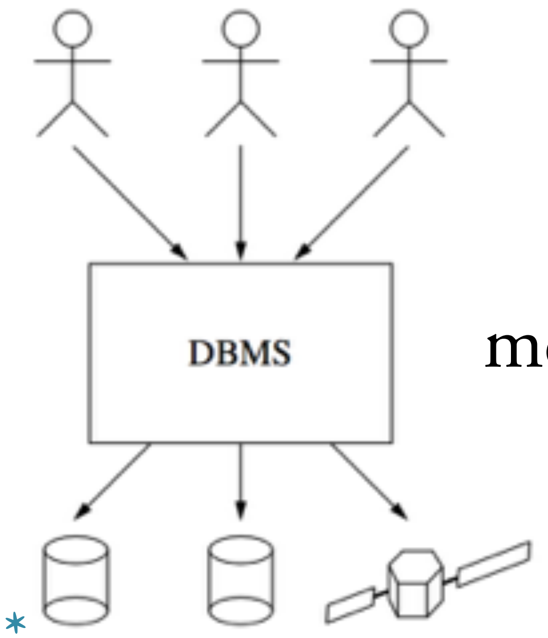
Conjunctive Queries

Acyclicity



Databases

DBMS = a collection of data, structured in some way + a way of defining, querying, updating the data inside



*

mediate between humans, processes & data

* [Abitebou, Hull, Vianu "Foundations of databases"]

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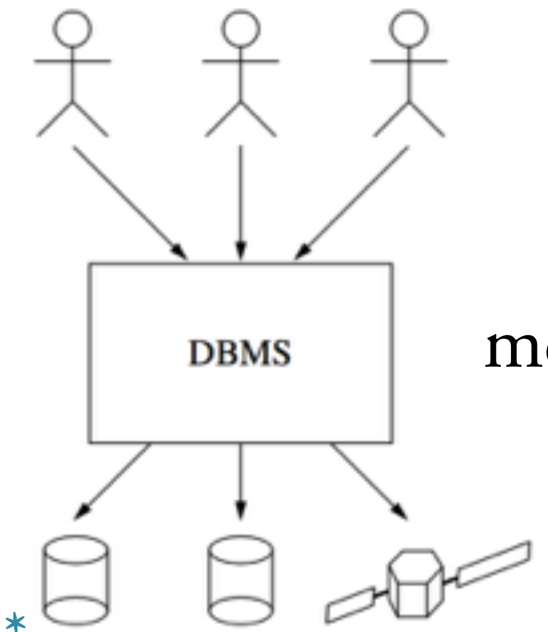

*

mediate between humans, processes & data

Data model

- how the data is logically organised
- mathematical abstraction for representing data
- independent from physical organisation

DBMS also implement: transactions, concurrency, access control, resiliency…

* [Abitebou, Hull, Vianu “Foundations of databases”]

Relational databases, historical outlook

1970–72: E.F. Codd (IBM San Jose research lab) introduces the "relational data model" and two query languages: "relational algebra" and "relational calculus"

1974–75: IBM researchers start implementing

- "System R": first relational database management system (RDBMS).
- SEQUEL: a query langauge based on relational algebra

1983: IBM "DB2" is released, based on System R.

And UC Berkley released Ingres RDBMS

1979: Oracle Corporation is founded

1981: Codd receives Turing award

Now: multi-billion industry

Company	2006 Revenue	2006 Market Share
Oracle	7.168B	47.1%
IBM	3.204B	21.1%
Microsoft	2.654B	17.4%
Teradata	494.2M	3.2%
Sybase	486.7M	3.2%
Other	1.2B	7.8%
Total	15.2B	100%

Relational databases

Relational data model = data logically organised into relations (“tables”).

What’s a **relation**?

- a (finite) subset of the cartesian product of sets
- a “table” with rows and columns

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like:

$$\{ (1,a,2), (2,b,6), (2,a,1) \} \subseteq N \times \{a,b\} \times N$$

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→ a "tuple" (a "3-tuple")

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like: "

1	a	2
2	b	6
2	a	1

"

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DB = A **schema**: names of tables and attributes

An **instance**: data conforming to the schema

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Films (Title:string, Director:string, Actor:string)

Schedule (Theatre:string, Title:string)

An instance: data conforming to the schema

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<u>Films</u> (Title:string, Director:string, Actor:string)

<u>Schedule</u> (Theatre:string, Title:string)

An **instance**: data conforming to the schema

Films

Title	Director	Actor
8 1/2	Fellini	Mastroianni
Shining	Kubrick	Nicholson
Dr. Strangelove	Kubrick	Sellers
8 femmes	Ozon	Ardant

Schedule

Theatre	Title
Utopia	Dr. Strangelove
Utopia	8 1/2
UGC	Dr. Strangelove
UGC	8 femmes

Relational databases

Relational data model = data logically organised into relations (“tables”).

We assume all elements come from
a fixed set of *constants* or *data values* **U**.

Relational databases: queries

What is a **query** *q* ?

A mapping that
- takes a database instance **D**
- returns a relation $q(\mathbf{D}) \subseteq \mathbf{U}^{\mathbf{r}}$ of fixed arity **r**

computable!

What is a **query** q ?

A mapping that
takes a database instance $\mathbf{D}$
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Relational databases: queries

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generic!
(order independent)

Relational databases: queries

computable!

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Boolean query: r=0
Either "yes" { () } or "no" { }

Relational databases: queries

What is a **query** *q* ?

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What do we care about queries?

expressive power

evaluation

static analysis

The fundamental questions:

How to query the relational data model?

How efficient/expressive is it?

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How to query the relational data model?

How efficient/expressive is it?

Query languages

Query Language

Syntax + **Semantics**

Syntax	Semantics
Expressions for querying the db, governed by syntactic rules	Interpretation of symbols in terms of some structure
"Select X from Y"	Retrieves all strings in column X of table Y
"y :- ∀x (x ≤ y)"	Returns the maximum element of the set.

Relational Algebra (RA)

[Codd, 1970]

Syntax: $E := R, S, \ldots \mid E \cup E \mid E \setminus E \mid E \times E \mid \pi_M(E) \mid \sigma_\Theta(E)$

where $M \subseteq \mathbb{N}$

$\Theta \subseteq \mathbb{N} \times \{=, \neq\} \times \mathbb{N}$

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- $\sigma_{\{i_1=j_1,\ldots,i_n=j_n\}}(R) \coloneqq \{(x_1, \ldots, x_m) \in R \mid (x_{i_1}=x_{j_1}) \wedge \cdots \wedge (x_{i_n}=x_{j_n})\}$: Selection

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$\sigma_{\{1=3, 1\neq 2\}}(\{(1,2,1), (2,2,2)\}) = \{(1,2,1)\}$

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Question 1: What is the RA expression for

$\{ (v_1, v_2) \mid$ there are $w_1 \neq w_2$ so that $(v_1, w_1) \in R_1$ and $(v_2, w_2) \in R_2 \}$?

Question 2: $\pi_2(\sigma_{1=3}(\pi_2(\sigma_{1=3}(R_1 \times R_2)) \times R_2)) =$?

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

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Answer: $\pi_{\{1,3\}}(\sigma_{1\neq 3}(R_1 \times R_2))$

a	b
b	a
c	a
c	b

Question 2: $\pi_2(\sigma_{1=3}(\pi_2(\sigma_{1=3}(R_1 \times R_2)) \times R_2)) = ?$

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
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a	b
b	a
c	a
c	b

Question 2: $\pi_2(\sigma_{1=3}(\pi_2(\sigma_{1=3}(R_1 \times R_2)) \times R_2)) =$?

Answer (only one element): b

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

RA = Basic SQL

no domain-specific features, aggregation, etc

```
Select X
From R1,…,Rn
Where Z
```

$\iff$ $\pi_X(\sigma_Z(R_1 \times \cdots \times R_n))$

… or … $\iff$ union

… not in (…) $\iff$ difference

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Select X
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… not in (…) $\Longleftrightarrow$ difference

$\pi_2(\sigma_{1\neq 3}(R_1 \times R_2)) \rightsquigarrow$

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

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```
Select X
From R₁,…,Rₙ
Where Z
```

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… not in (…) $\Longleftrightarrow$ difference

$\pi_2(\sigma_{1 \neq 3}(R_1 \times R_2)) \rightsquigarrow$

```
Select R₁.2 as foo
From R₁, R₂
Where R₁.1 ≠ R₂.1
```

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
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$\pi_2 (\sigma_{1\neq 3}(R_1 \times R_2))$ $\rightsquigarrow$

```
Select R1.2 as foo
From R1, R2
Where R1.1 ≠ R2.1
```

❄

$\pi_2(\sigma_{1=3}($ ❄ $\times R_2))$ $\rightsquigarrow$

R_1

a	3
b	2
c	4
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R_2

a	4
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From R₁, R₂
Where R₁.1 ≠ R₂.1
```

★

❄

R₁

a	3
b	2
c	4
b	3
a	2

R₂

a	4
b	1
b	2
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b	3

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```
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$\underbrace{\pi_2 (\sigma_{1\neq 3}(R_1 \times R_2))}_{❄} \rightsquigarrow$

```
Select R₁.2 as foo
From R₁, R₂
Where R₁.1 ≠ R₂.1
```
) ★

$\pi_2(\sigma_{1=3}(❄ \times R_2)) \rightsquigarrow$

```
Select foo
From ★, R₂
Where foo = R₂.2
```

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

Denotational languages

Procedural

Algebra $\rightsquigarrow$ *How* to obtain the result

Logics $\rightsquigarrow$ *What* is the property of the result

Declarative

Denotational languages

Relational Algebra

operations on tables

Algebra $\leadsto$ *How* to obtain the result

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Logics $\leadsto$ *What* is the property of the result

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Denotational languages

Relational Algebra
operations on tables

Algebra $\rightsquigarrow$ *How* to obtain the result

Procedural

Logics $\rightsquigarrow$ *What* is the property of the result

Declarative

First Order logic
properties on mathematical structures

Denotational languages

Relational Algebra
operations on tables

Algebra ⇝ ***How*** to obtain the result — Procedural

Logics ⇝ ***What*** is the property of the result — Declarative

First Order logic
properties on mathematical structures

FO = First-Order logic

Relational structures

A **structure** is:

$$A = (D, R_1, \ldots, R_n, f_1, \ldots f_n)$$

D is a non-empty set, the domain

R_i is an m-ary relation for some m (ie, $R_i \subseteq D^m$)

f_i is an n-ary function for some n (ie, $f_i : D^n \longrightarrow D$)

Relational structures

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A **graph** G = (V,E)

- V: nodes
- E ⊆ V^2: edges (binary relation)
- (no functions)

A **group**, like $(\mathbb{N},+)$

- $\mathbb{N}$: natural numbers
- (no relations)
- $+: \mathbb{N}^2 \longrightarrow \mathbb{N}$ addition (binary function)

First-order logic

First-order logic

FO

variables x, y, z, …

quantifiers: ∃,∀

Boolean connectives: ¬, ∧, ∨

A language to talk about **structures**

Variables range over the **domain**

Atomic formulas: $R(x_1, \ldots, x_m)$, $x{=}y$

First-order logic

FO

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Language to talk about **graphs**

Variables range over **nodes**

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First-order logic

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A language to talk about **structures**

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Atomic formulas: $R(x_1, \ldots, x_m)$, $x=y$

A **graph** G = (V,E)

- V: nodes
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Language to talk about **graphs**

Variables range over **nodes**

Atomic formulas: E(x,y), x = y

Formulas: Atomic formulas + connectives + quantifiers

“The node x has at least two neighbours”

$$\exists y\ \exists z\ (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$$

"The node x has at least two neighbours"

$\varphi(x) = \exists y\, \exists z\, (\neg(y{=}z) \wedge E(x,y) \wedge E(x,z))$

↳ free

x is free = not quantified

(a property of a node in the graph)

"The node x has at least two neighbours"

$\varphi(x) = \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

↳ free

x is free = not quantified

(a property of a node in the graph)

"Each node has at least two neighbours"

$\forall x\, \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

"The node x has at least two neighbours"

$\varphi(x) = \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

↳ free

x is free = not quantified

(a property of a node in the graph)

"Each node has at least two neighbours"

$\psi = \forall x\, \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

the formula is a sentence

= no free variables

(a property of the graph)

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Question:

- How to express in FO

  "*Every two adjacent nodes have a common neighbour*" ?

- Does it have free variables? Is it a sentence?

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Question: • How to express in FO

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Answer: $\forall x \forall y \left(\neg E(x,y) \vee \exists z \left(\left(E(x,z) \vee E(z,x)\right) \wedge \left(E(y,z) \vee E(z,y)\right) \right) \right)$

Binding

To evaluate a formula ϕ we need a graph $G=(V,E)$ and a **binding** α that maps free variables of ϕ to nodes of G.

$G \models_{\alpha} \phi(x_1,\dots,x_n)$ $\alpha : \{x_1,\dots,x_n\} \longrightarrow V$ assigns nodes to free variables

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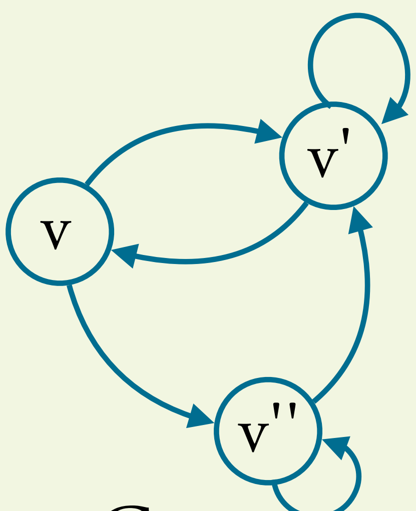

G

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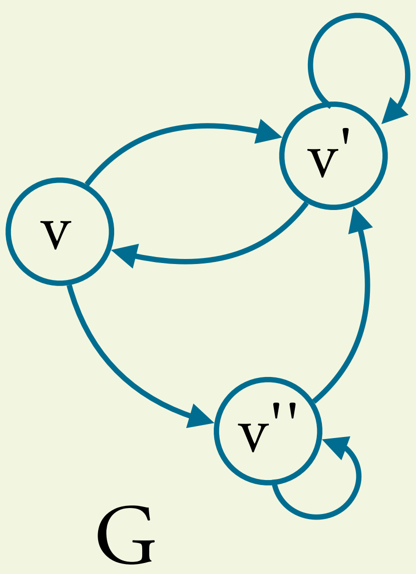

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$G \models_\alpha x=y$ iff $\alpha(x)=\alpha(y)$

$G \models_\alpha E(x,y)$ iff $(\alpha(x),\alpha(y)) \in E$

Formulas as queries

$\phi(x_1, \ldots, x_n)$ evaluated on $G=(V,E)$ yields all the bindings that satisfy ϕ:

$$\phi(G) = \{ \, (\alpha(x_1),\ldots,\alpha(x_n)) \mid G \models_\alpha \phi, \;\; \alpha: \{x_1,\ldots,x_n\} \longrightarrow V \, \}$$

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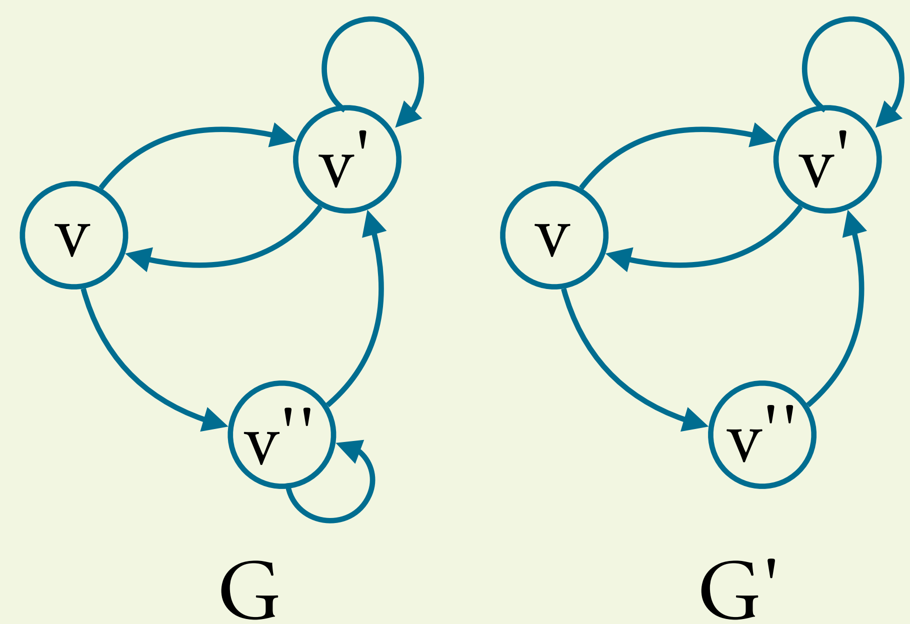

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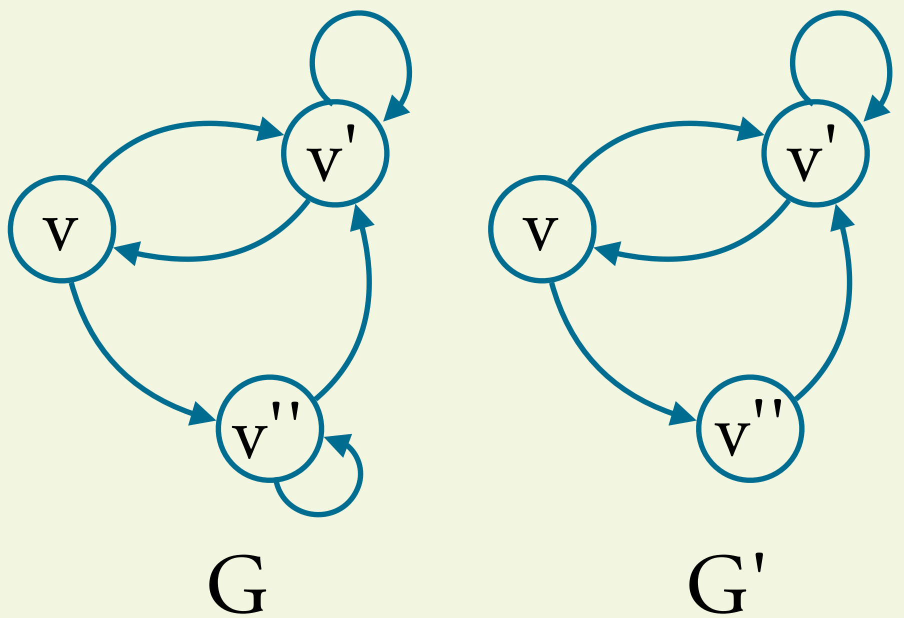


"Every node has two neighbours"

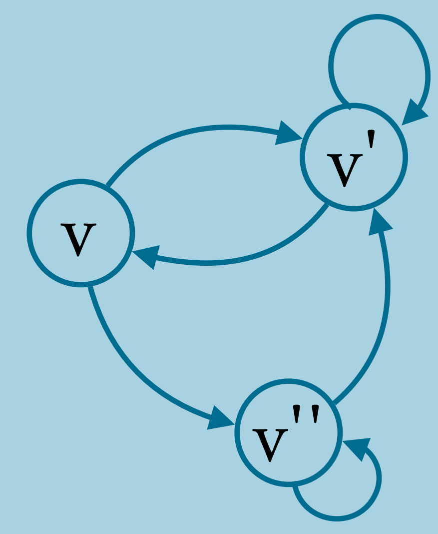
$\psi = \forall x\, \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

$\psi(G) = \{()\}$ ⤳ set with one element: the 0-tuple

$\psi(G') = \{\}$ ⤳ empty set

Question: Which bindings α verify $G \models_\alpha \phi$ for

$$\phi(x,y) = \exists z\, (E(x,z) \wedge E(z,y))$$

and G =  ?

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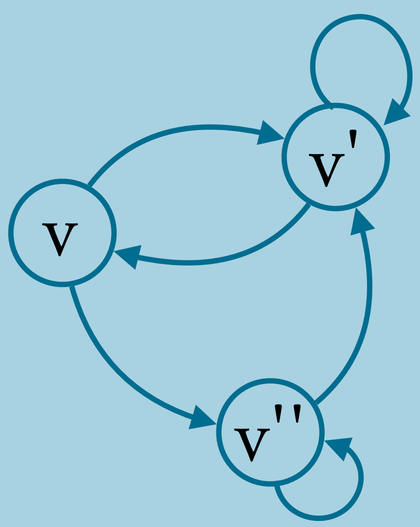
and $G =$  ?

Answer:

- $\alpha = \{\, x \mapsto v,\ y \mapsto v' \,\}$,
- $\alpha = \{\, x \mapsto v,\ y \mapsto v \,\}$,
- $\alpha = \{\, x \mapsto v',\ y \mapsto v' \,\}$,
- ... and all the rest

$$\phi(G) = \{v, v', v''\} \times \{v, v', v''\}$$

Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

Tables = Relations

Queries = Formulas

Rows = Tuples

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Like which?

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Finite model theory

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[E.F. Codd 1972]

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$$\text{RA} =^{*} \text{FO}$$

$$\textit{How} = \textit{What}$$

[E.F. Codd 1972]

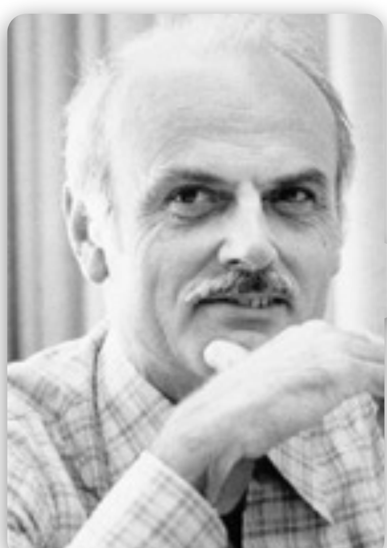
Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

Tables	=	Relations
Rows	=	Tuples
Queries	=	Formulas

RA =* FO

How = *What*



RA and FO logic have roughly* the same expressive power!

*FO without functions, with equality, on finite domains, …

Formulas as queries

RA ⊆ FO

- $R_1 \times R_2 \quad \rightsquigarrow \quad R_1(x_1, \ldots, x_n) \land R_2(x_{n+1}, \ldots, x_m)$
- $R_1 \cup R_2 \quad \rightsquigarrow \quad R_1(x_1, \ldots, x_n) \lor R_2(x_1, \ldots, x_n)$
- $\sigma_{\{i_1=j_1,\ldots,i_n=j_n\}}(R) \quad \rightsquigarrow \quad R(x_1, \ldots, x_m) \land (x_{i_1}=x_{j_1}) \land \cdots \land (x_{i_n}=x_{j_n})$
- $\pi_{\{i_1,\ldots,i_n\}}(R) \quad \rightsquigarrow \quad \exists(\{x_1,\ldots,x_m\} \setminus \{x_{i_1},\ldots,x_{i_n}\}).\ R(x_1, \ldots, x_m)$
- $R_1 \setminus R_2 \quad \rightsquigarrow \quad R_1(x_1, \ldots, x_n) \land \neg R_2(x_1, \ldots, x_n)$
- ...

Formulas as queries

$\boxed{\text{FO} \subseteq \text{RA}}$ does not hold in general!

Formulas as queries

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"the complement of R" $\notin$ RA

$\in$ FO : $\neg R(x)$

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⇝ We restrict variables to range over **active domain**

Formulas as queries

FO $\not\subseteq$ RA

"the complement of R" $\notin$ RA
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⇝ We restrict variables to range over **active domain** → elements in the relations

FO^{act}
=
FO restricted
to active domain

Formulas as queries

FO $\not\subseteq$ RA

"the complement of R" $\notin$ RA

$\in$ FO : $\neg R(x)$

⇝ We restrict variables to range over **active domain** ⋯▸ elements in the relations

FO^{act}

=

FO restricted to active domain

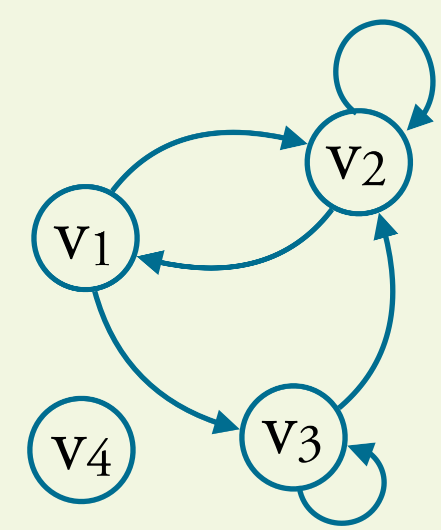
$\phi_1(x) = \forall y\, E(y,x)$

$\phi_1(G) = \{v_2\}$

$\phi_2(x,y) = \neg E(x,y)$

$\phi_2(G) = \{(v_1,v_1),(v_3,v_1),(v_2,v_3)\}$

G =



First-order logic restricted to active domain

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$G \models_\alpha \exists x\, \phi$ iff for **some** $v \in ACT(G)$ and $\alpha' = \alpha \cup \{x \mapsto v\}$ we have $G \models_{\alpha'} \phi$

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$G \models_\alpha x=y$ iff $\alpha(x)=\alpha(y)$

$G \models_\alpha E(x,y)$ iff $(\alpha(x),\alpha(y)) \in E$

$ACT(G) = \{v \mid$ **for some** v': $(v,v') \in E$ or $(v',v) \in E\}$

First-order logic restricted to active domain

$\mathrm{FO}^{\mathrm{act}} \subseteq \mathrm{RA}$

First-order logic restricted to active domain

$FO^{act} \subseteq RA$

Assume:

1. ϕ has variables $x_1,\dots,x_n$,
2. ϕ in normal form: $(\exists^* (\neg\exists)^*)^*$ + quantifier-free $\psi(x_1,\dots,x_n)$

$$\exists x_1 \, \exists x_2 \, \neg\exists x_3 \, \exists x_4 \, . \, (\, E(x_1,x_3) \wedge \neg E(x_4,x_2) \,) \vee (x_1 = x_3)$$

First-order logic restricted to active domain

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Adom = RA expression for active domain = "$\pi_1(E) \cup \pi_2(E)$"

Translation

- $(R(x_{i_1},\ldots,x_{i_n}))^+ \rightsquigarrow R$
- $(\, \exists x_i\, \phi(x_{i_1},\ldots,x_{i_n})\,)^+ \rightsquigarrow \pi_{\{i_1,\ldots,i_n\}\setminus\{i\}}(\, \phi^+\,)$
- $(x_i = x_j)^+ \rightsquigarrow \sigma_{\{i=j\}}(\, \mathbf{Adom} \times \cdots \times \mathbf{Adom}\,)$
- $(\psi_1(x_{i_1},\ldots,x_{i_n}) \wedge \psi_2(x_{i_1},\ldots,x_{i_n}))^+ \rightsquigarrow \psi_1^+ \cap \psi_2^+$
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First-order logic restricted to active domain

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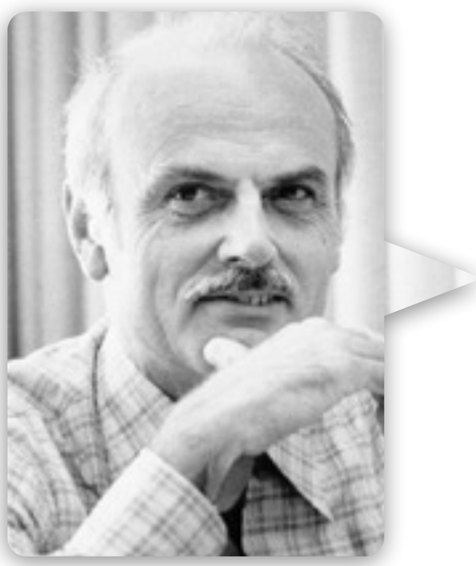
$A \cap B = (A \cup B) \setminus A \setminus B$

Corollary



$\mathbf{FO}^{\text{act}}$ is equivalent to RA

Question 1: How is $\pi_2(\sigma_{1=3}(\pi_2(\sigma_{1=3}(R_1 \times R_2)) \times R_2))$ expressed in FO?
Remember: R_1,R_2 are binary

Question 2: How is $\exists y,z \,.\, (R_1(x,y) \wedge R_1(y,z) \wedge x \neq z)$ expressed in RA?
Remember: The signature is the same as before (R_1,R_2 binary)

- $R_1 \cup R_2$
- $R_1 \times R_2$
- $R_1 \setminus R_2$
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) := \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$
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Answer: $\exists x_2 . \Big(\exists x_1,x_4 . \big(R_1(x_1,x_2) \wedge R_2(x_1,x_4)\big) \wedge R_2(x_2,x_5) \Big)$

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Answer: $\pi_1(\sigma_{\{2=3, 1\neq 4\}}(R_1 \times R_1))$

FO = RA = SQL

Logic | Algebra | Programming language

over active domain

on finite domains

very basic

FO = RA = SQL

Logic

Algebra

Programming language

Algorithmic problems for query languages

Evaluation problem: Given a query **Q**, a database instance **db**, and a tuple **t**, is **t** ∈ **Q**(**db**) ?

⇝ How hard is it to retrieve data?

Algorithmic problems for query languages

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Emptiness problem: Given a query **Q**, is there a database instance **db** so that **Q**(**db**) ≠ ∅ ?

⇝ Does Q make sense? Is it a contradiction? (Query optimization)

Algorithmic problems for query languages

Evaluation problem: Given a query **Q**, a database instance **db**, and a tuple **t**, is $\mathbf{t} \in \mathbf{Q}(\mathbf{db})$?

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Emptiness problem: Given a query **Q**, is there a database instance **db** so that $\mathbf{Q}(\mathbf{db}) \neq \varnothing$?

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Equivalence problem: Given queries $\mathbf{Q}_1$, $\mathbf{Q}_2$, is

$$\mathbf{Q}_1(\mathbf{db}) = \mathbf{Q}_2(\mathbf{db})$$

for all database instances **db**?

⇝ Can we safely replace a query with another? (Query optimization)

Complexity theory

What can be **mechanized**? $\rightsquigarrow$ decidable/undecidable

How **hard** is it to mechanise? $\rightsquigarrow$ complexity classes

Complexity theory

Domino H's 10th PCP K ⋮

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usage of resources:
- time
- memory

Complexity theory

Domino, H's 10th, PCP, K, ⋮

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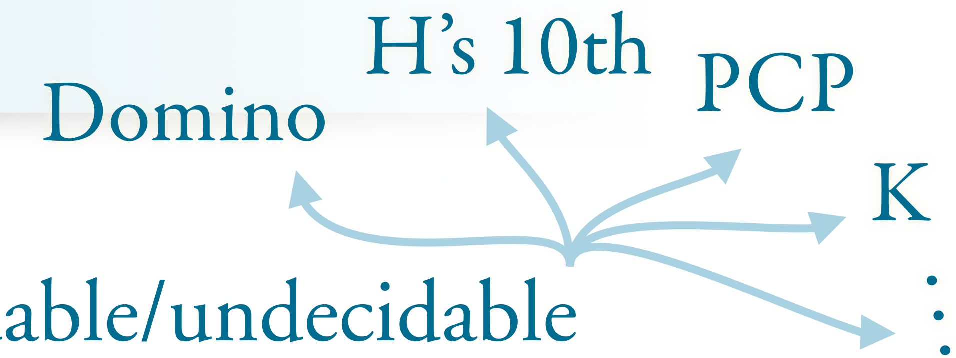
Algorithm Alg is TIME-bounded by a function $f: \mathbb{N} \longrightarrow \mathbb{N}$ if Alg(*input*) uses less than $f(|input|)$ units of TIME.

Complexity theory



What can be **mechanized**? $\rightsquigarrow$ decidable/undecidable

How **hard** is it to mechanise? $\rightsquigarrow$ complexity classes

usage of resources:
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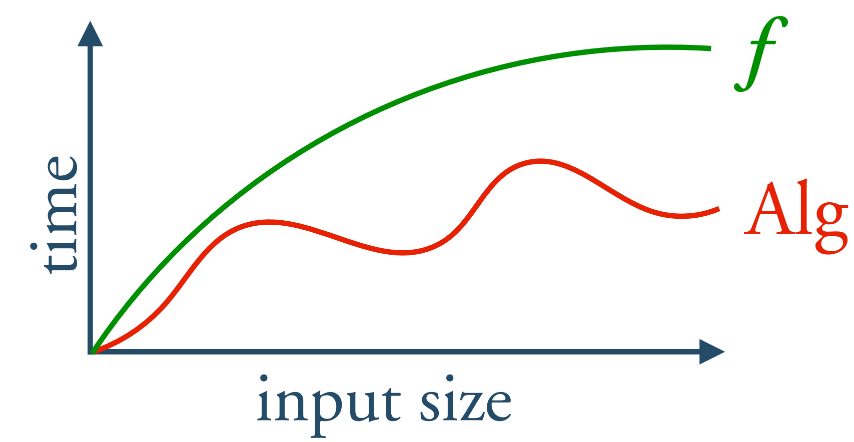

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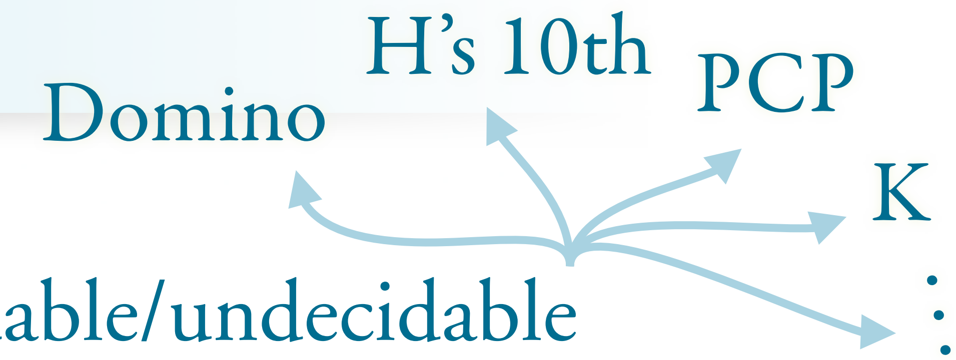

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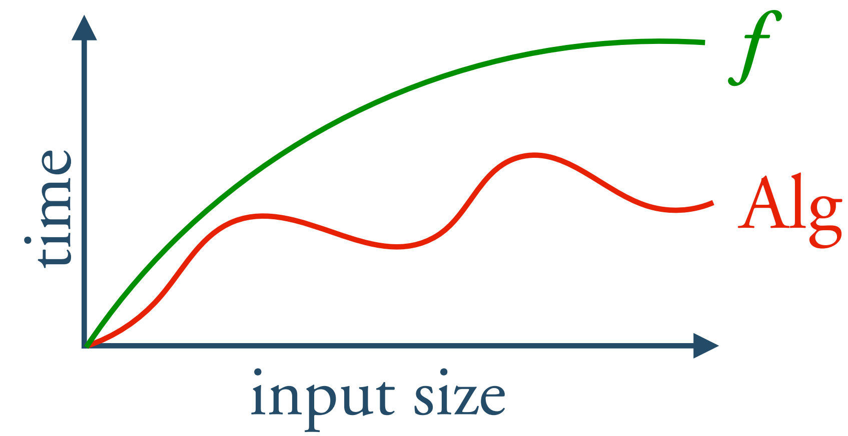

Complexity theory

Domino, H's 10th, PCP, K, ⋮

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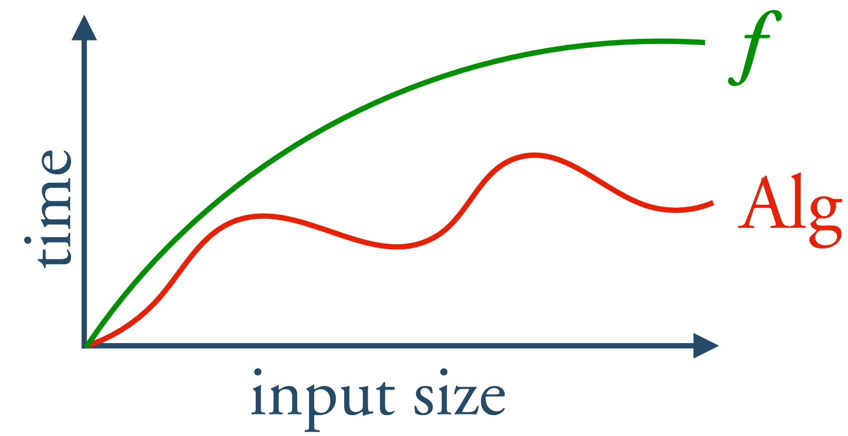

$$\mathrm{LOGSPACE} \subsetneq \mathrm{PTIME} \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXPTIME} \subseteq \cdots$$

Complexity theory

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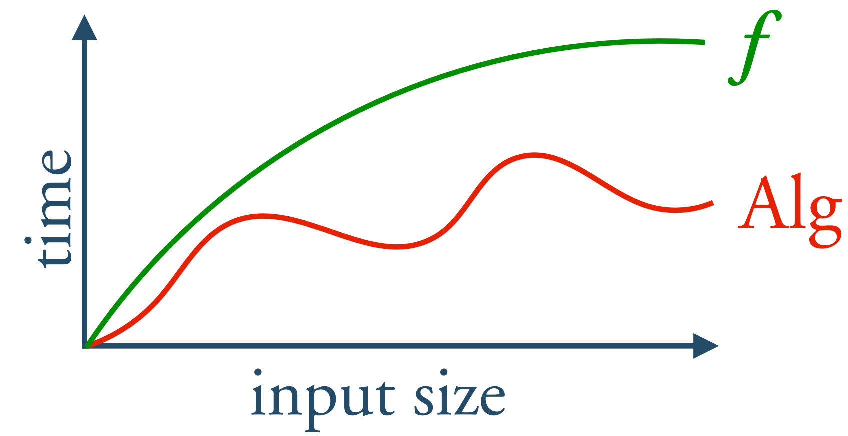

LOGSPACE ⊊ PTIME ⊆ PSPACE ⊆ EXPTIME ⊆ . . .

- LOGSPACE: SPACE-bounded by log(n)
- PTIME: TIME-bounded by a polynomial
- PSPACE: SPACE-bounded by a polynomial

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, \ldots, x_n)$, a graph G, and a binding α, does $G \models_\alpha \phi$?

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DECIDABLE ⇝ foundations of the database industry

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DECIDABLE $\rightsquigarrow$ foundations of the database industry

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Proof: By reduction from the Domino (aka Tiling) problem.

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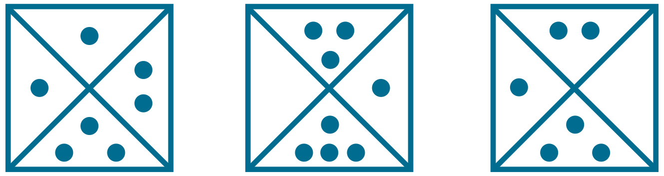
Reduction from P to P': Algorithm that solves P using a O(1) procedure
" P'(x) "
that returns the truth value of P'(x).

The (undecidable) Domino problem

Domino

Input: 4-sided dominos: 

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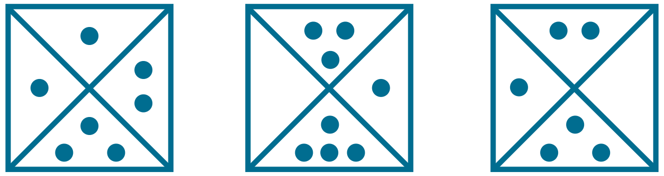
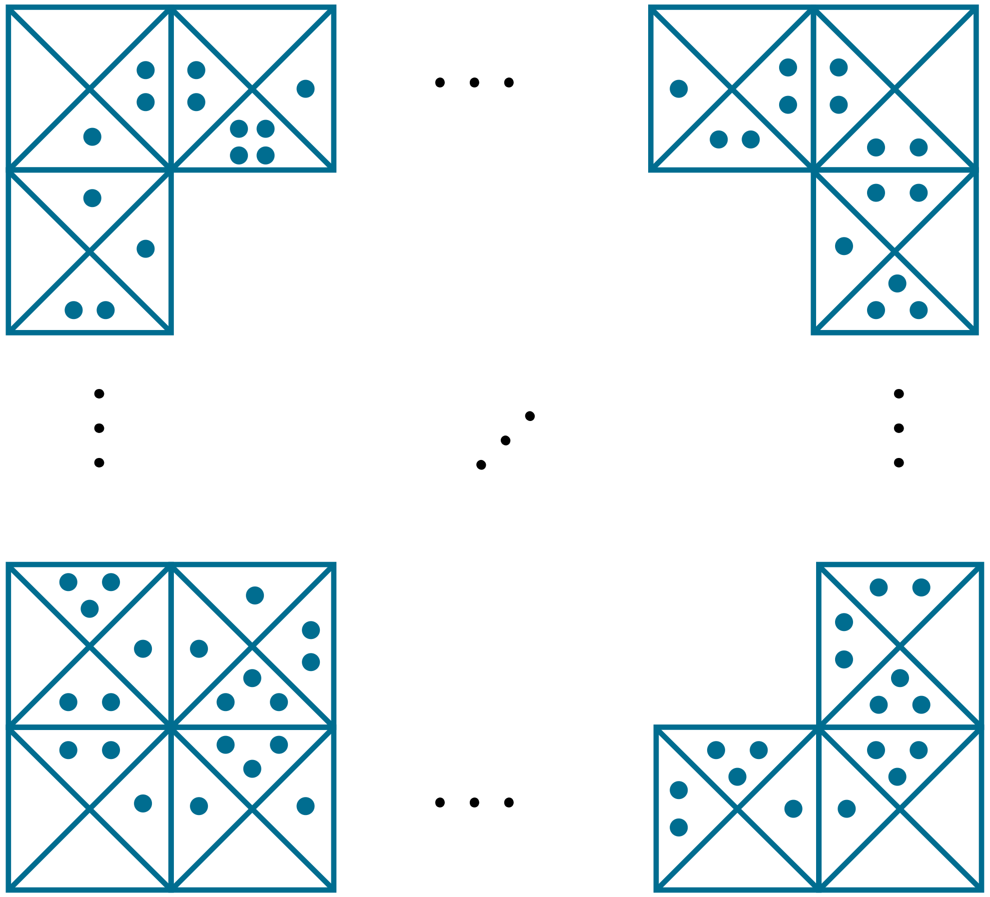
Input: 4-sided dominos: 

Output: Is it possible to form a white-bordered rectangle? (of any size)



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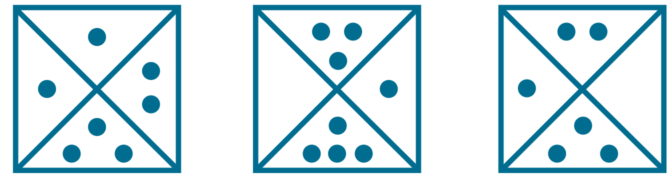
Input: 4-sided dominos: 

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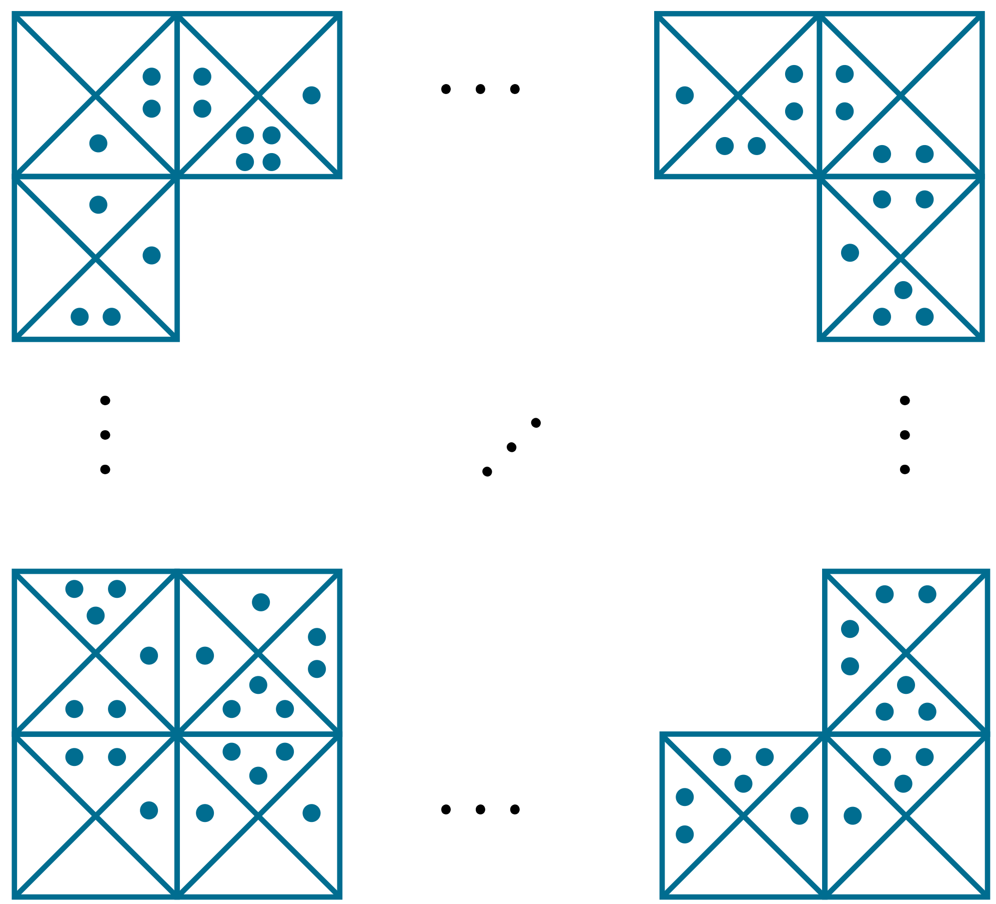
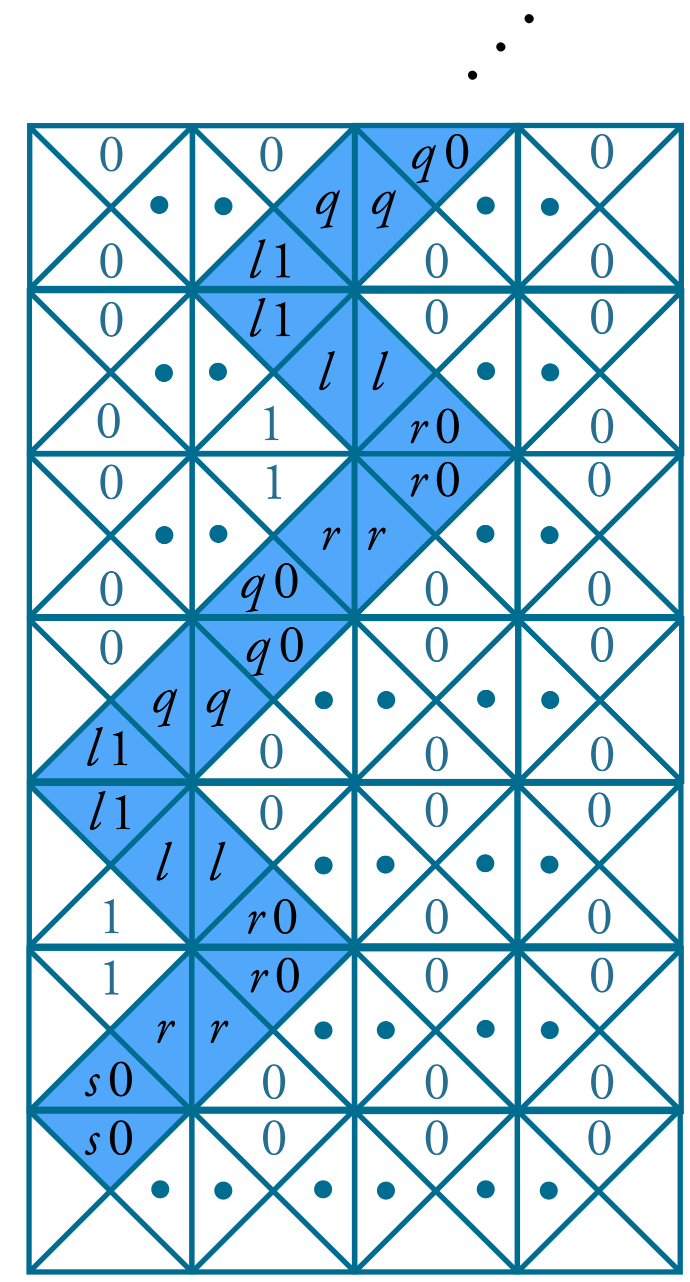


Rules: sides must match,
you can't rotate the dominos, but you can 'clone' them.

Domino - Why is it undecidable?

It can easily encode *halting* computations of Turing machines:



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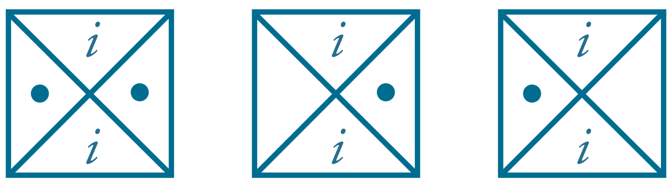

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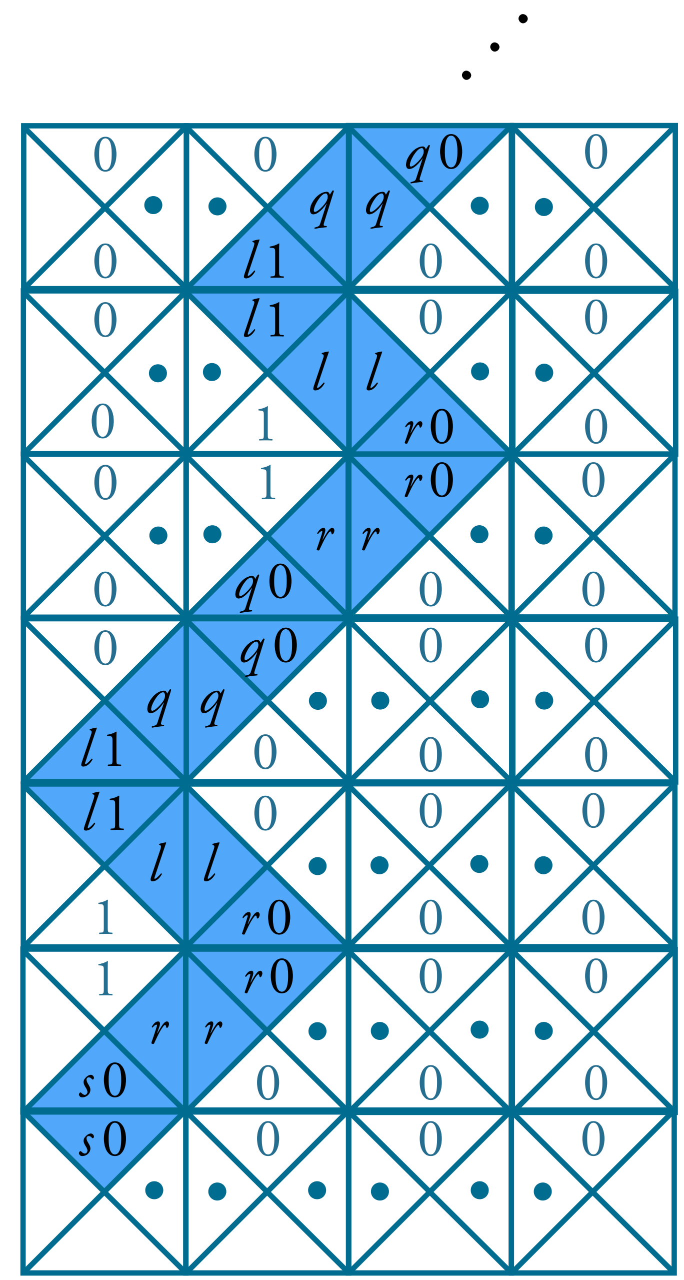

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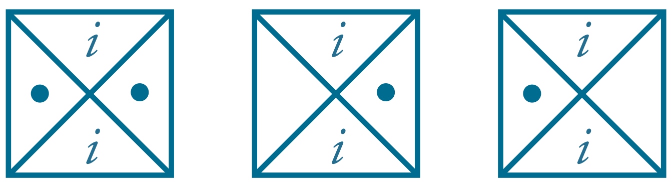

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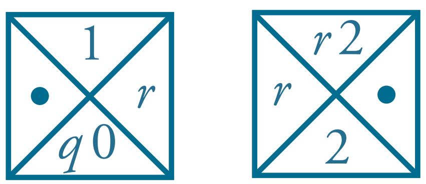

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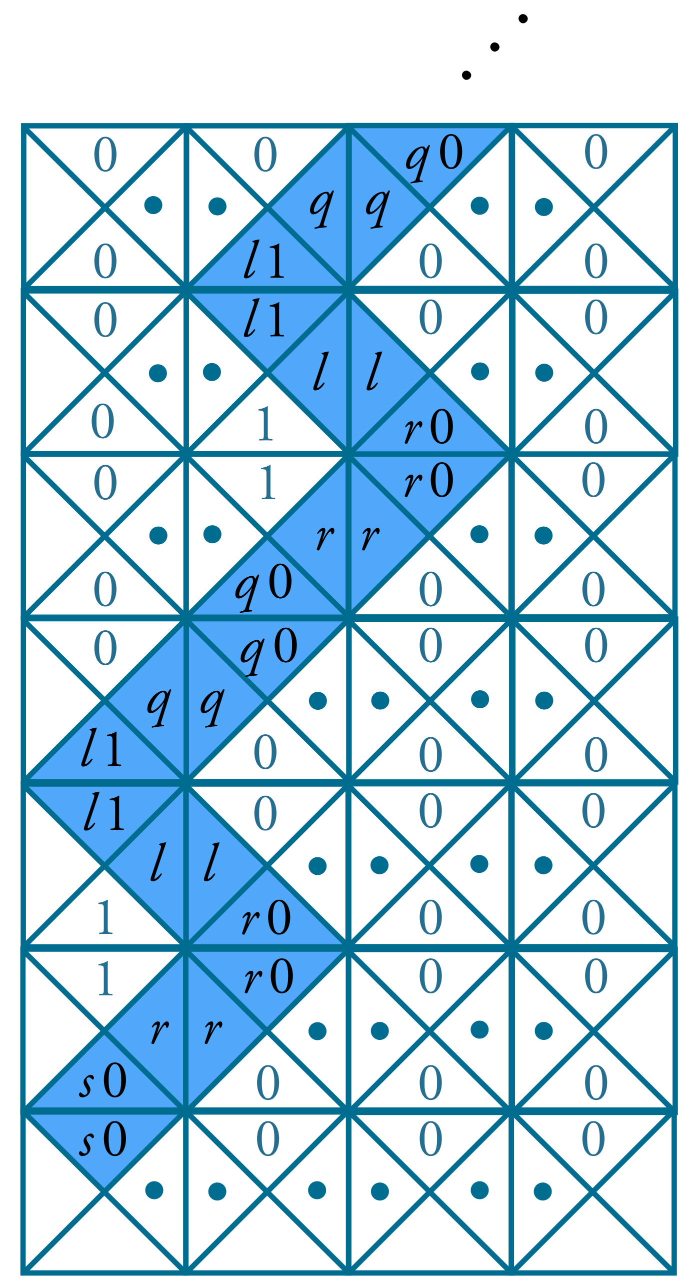

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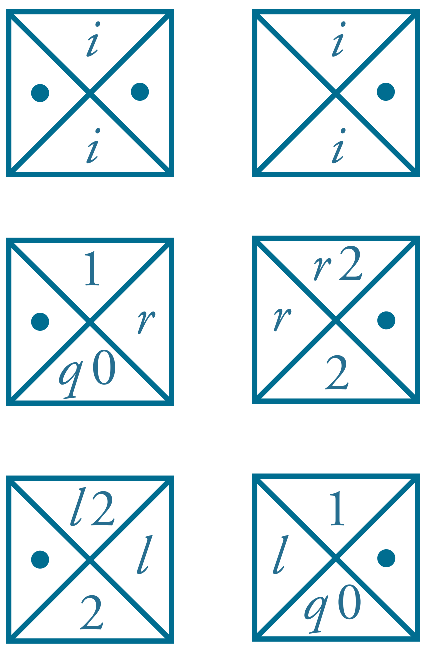

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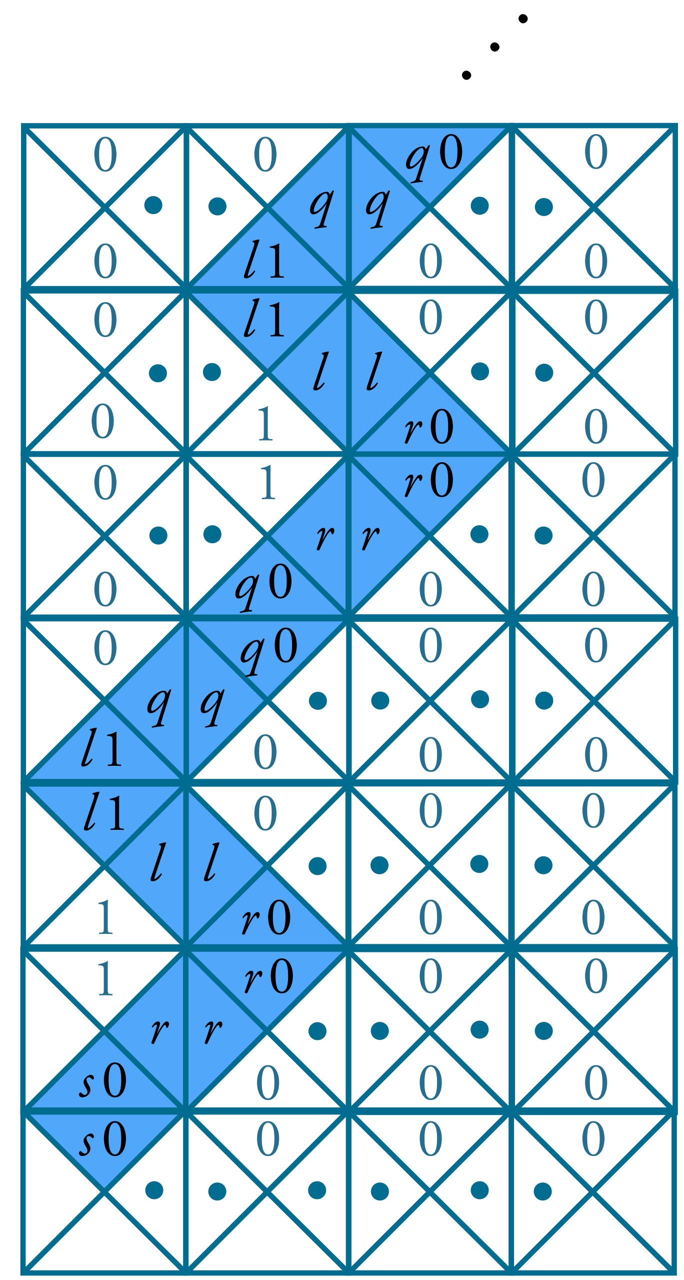

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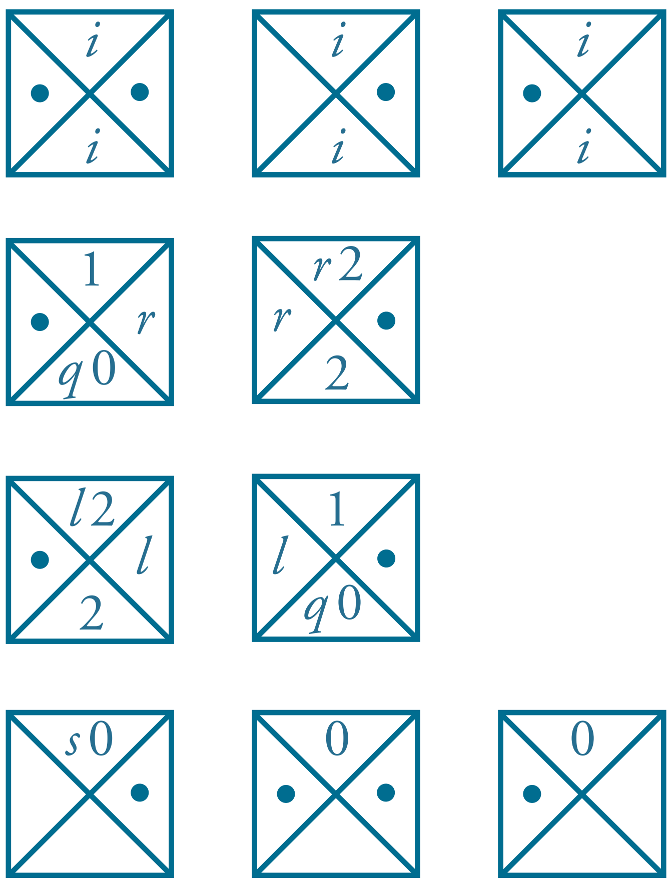

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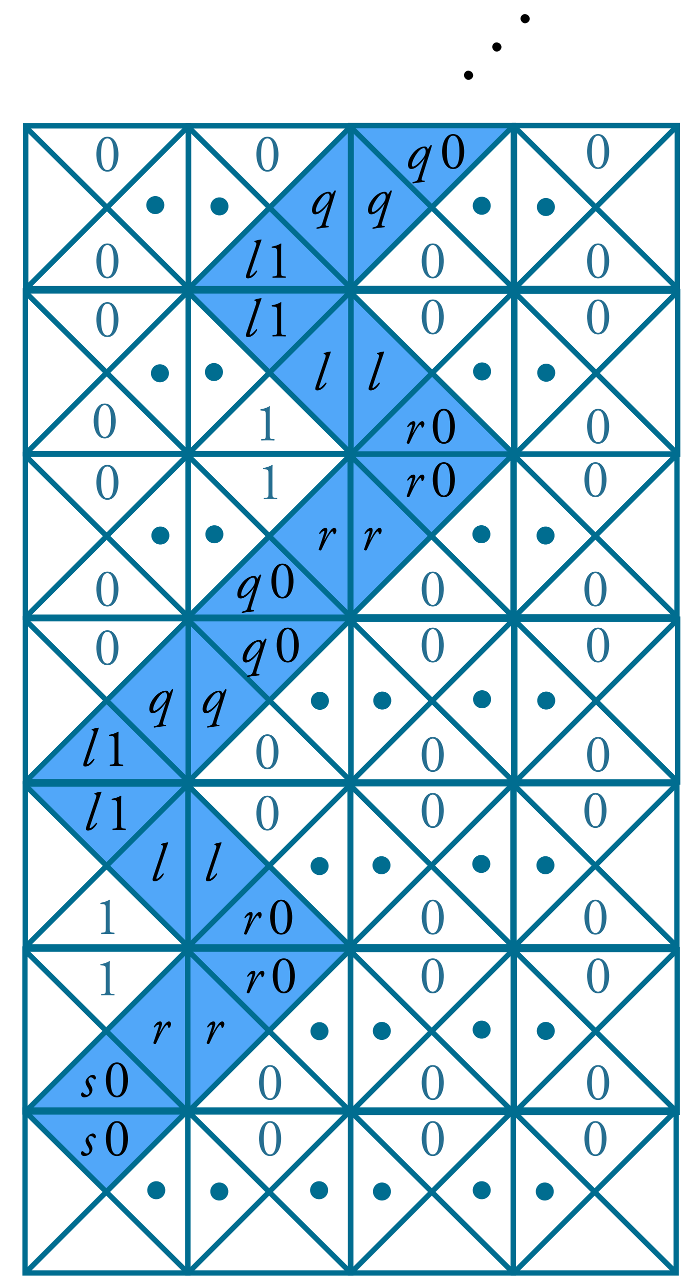

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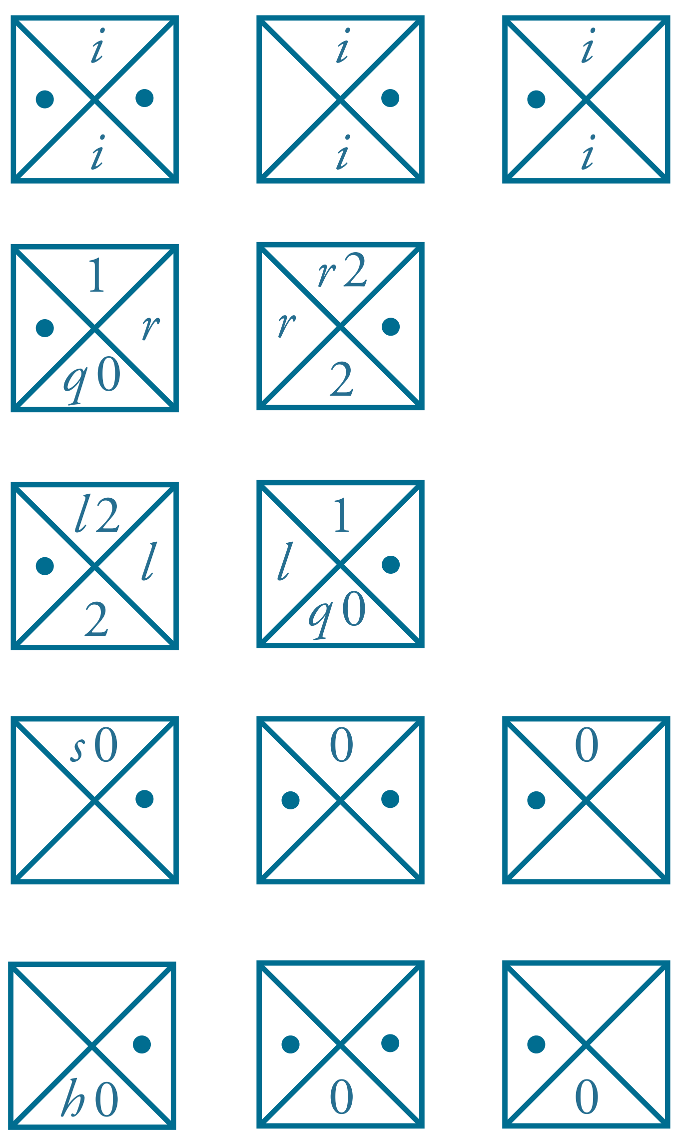

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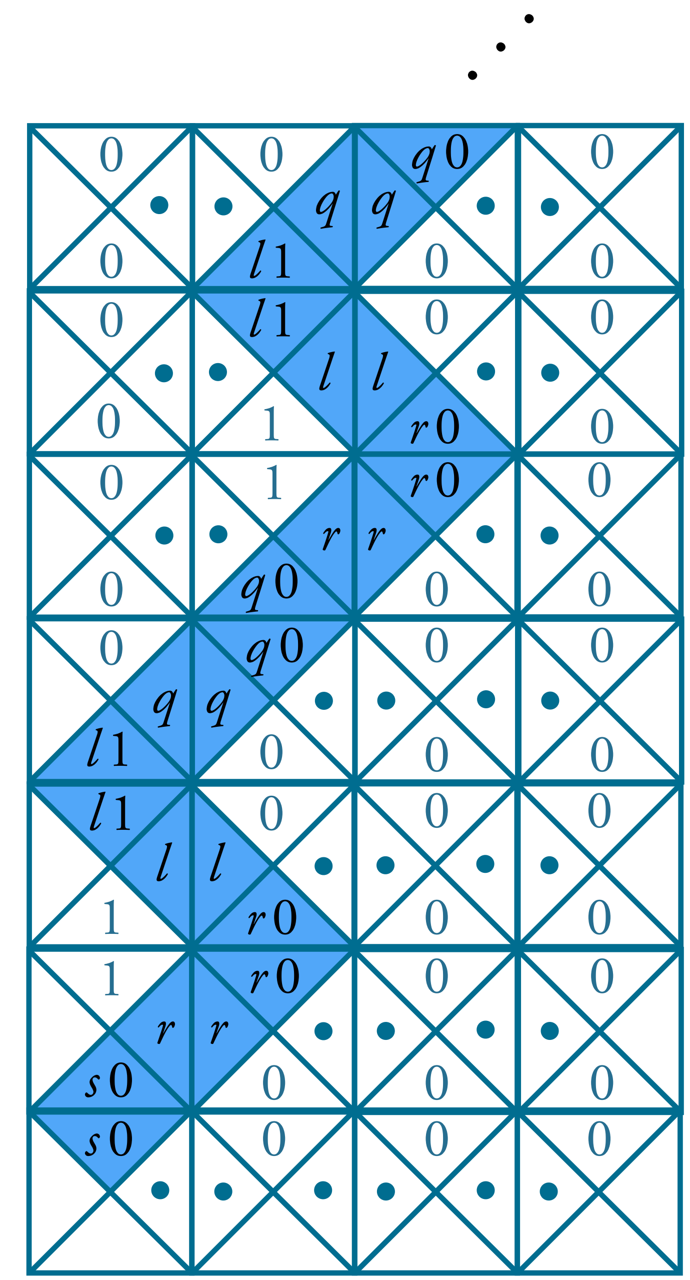

Domino ⇝ Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** H(,) and V(,) are relations representing bijections such that…

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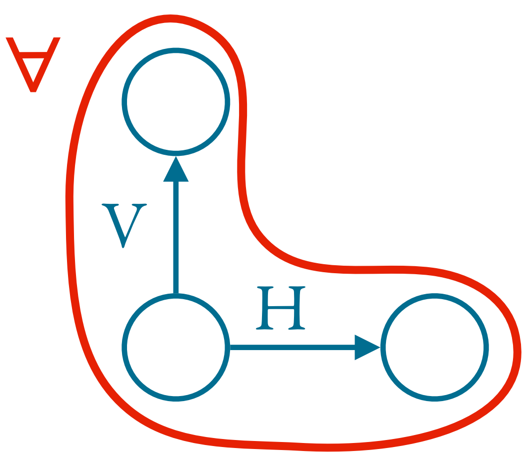

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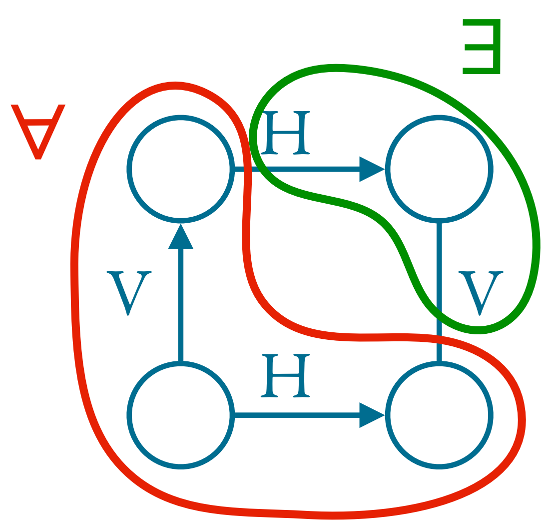

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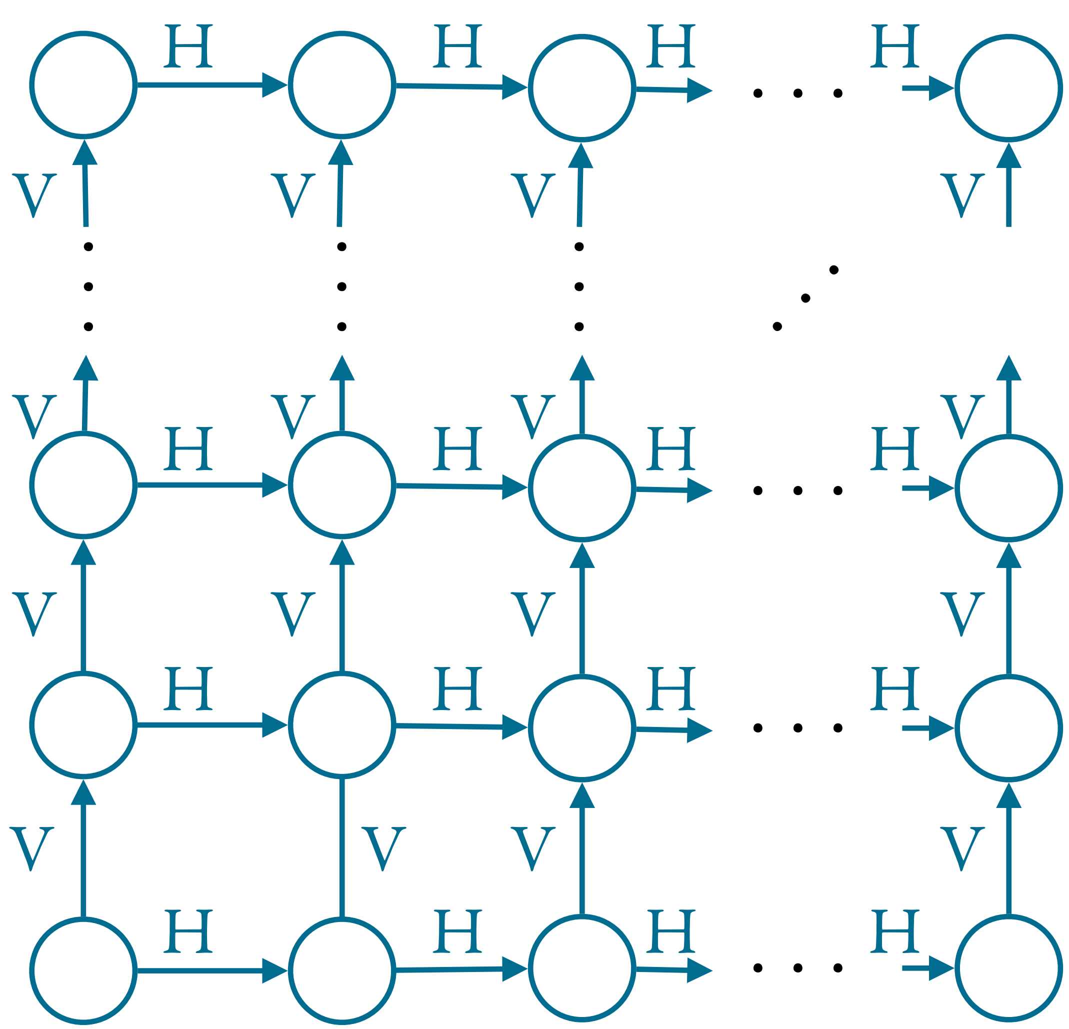

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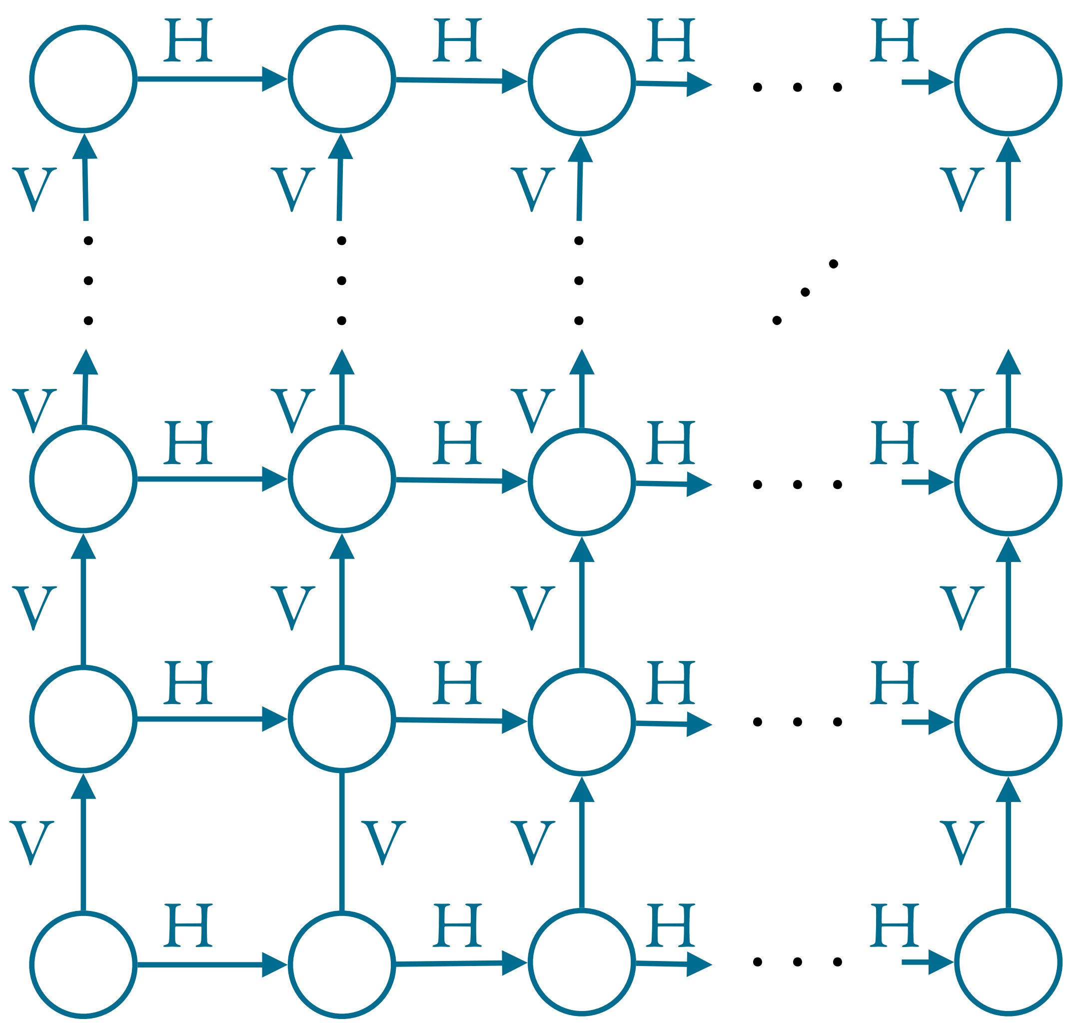

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a unary relation

$$D_{\boxtimes}(x)$$

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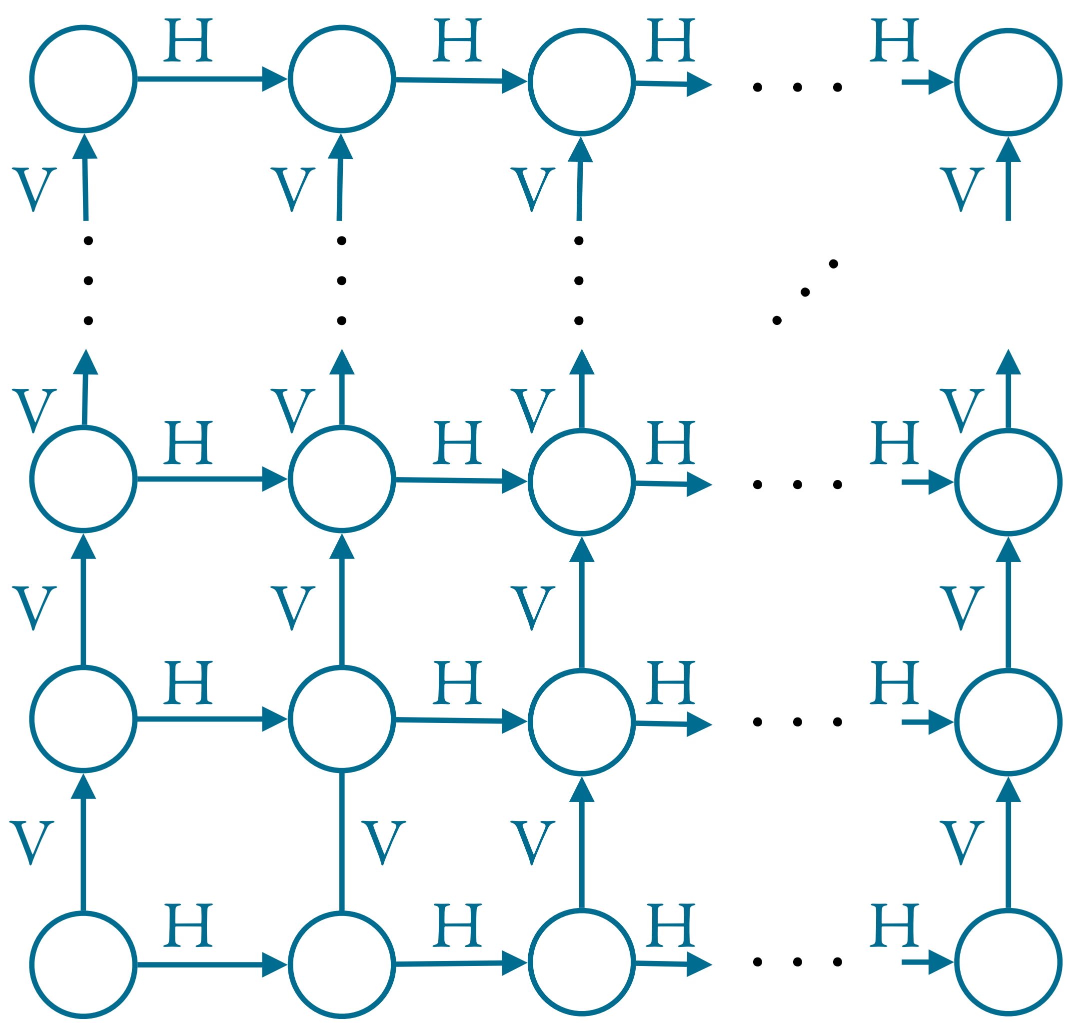

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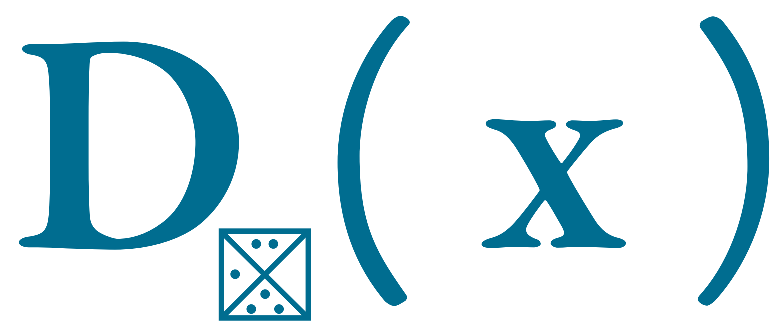
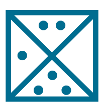


for each domino 

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if H(x,y), then $D_a(x) \wedge D_b(y)$

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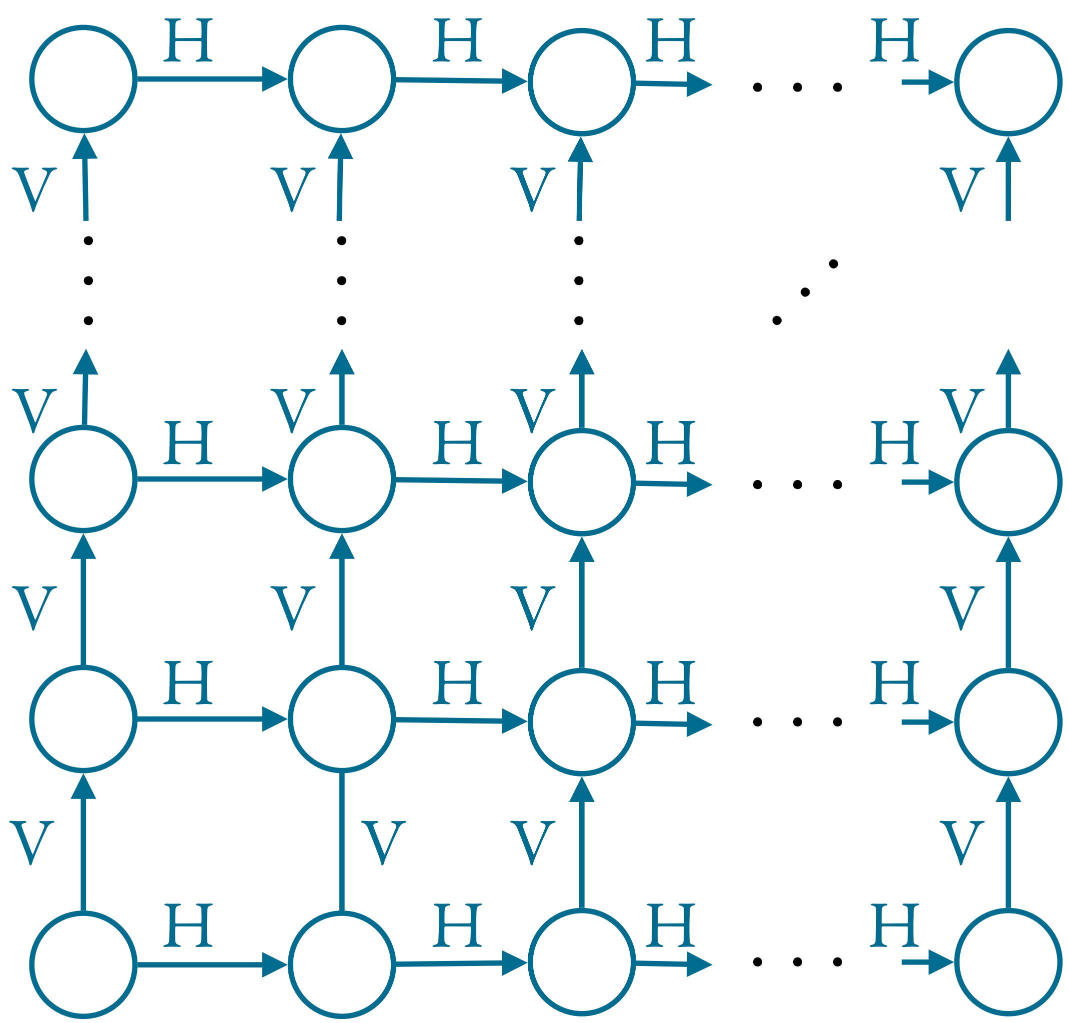

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4. **Borders are white.**

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, ..., x_n)$, a graph G, and a binding α, does $G \vDash_\alpha \phi$?

DECIDABLE ⇝ foundations of the database industry

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ϕ is **satisfiable** iff ϕ is **not equivalent** to $\bot$

Satisfiability problem undecidable ⇝ Equivalence problem undecidable

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Satisfiability problem undecidable ⇝ Equivalence problem undecidable

Actually, there are reductions in both senses:

$\phi(x_1,\dots,x_n)$ and $\psi(y_1,\dots,y_m)$ are **equivalent** iff

- $n=m$
- $(x_1=y_1) \wedge \cdots \wedge (x_n=y_n) \wedge \phi(x_1,\dots,x_n) \wedge \neg\psi(y_1,\dots,y_n)$ is unsatisfiable
- $(x_1=y_1) \wedge \cdots \wedge (x_n=y_n) \wedge \psi(x_1,\dots,x_n) \wedge \neg\phi(y_1,\dots,y_n)$ is unsatisfiable

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Evaluation problem for FO

Input:

$$\left(\begin{array}{l} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{array} \right.$$

Output: $G \models_\alpha \phi$?

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Encoding of $G = (V, E)$

- each node is coded with a bit string of size $\log(|V|)$,
- edge set is encoded by its tuples, e.g. (100,101), (010, 010), ...

Cost of coding: $||G|| = |E| \cdot 2 \cdot \log(|V|) \approx |V|$ (mod a polynomial)

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Encoding of $\alpha = \{x_1,\ldots,x_n\} \longrightarrow V$

- each node is coded with a bit string of size $\log(|V|)$,

Cost of coding: $||\alpha|| = n \cdot \log(|V|)$

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Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$

- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
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- If $\phi(x_1,\ldots,x_n) = \exists y \,.\, \psi(x_1,\ldots,x_n,y)$:
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Question:
How much space
does it take?

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  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$

  use 4 pointers $\rightsquigarrow$ LOGSPACE

- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$

- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
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- If $\phi(x_1,\ldots,x_n) = \exists y . \psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.

Question:
How much space
does it take?

Evaluation problem for FO

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$ Output: $G \models_\alpha \phi$?

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Question:
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Evaluation problem for FO

Input: $\left(\begin{array}{l} \phi(x_1,\dots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\dots,x_n\} \longrightarrow V \end{array} \right.$ Output: $G \models_\alpha \phi$?

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Question:
How much space does it take?

$\underbrace{2 \cdot \log(|G|) + \cdots + 2 \cdot \log(|G|)}_{\leq |\phi| \text{ times}} + k \cdot \log(|\alpha|+|G|)$ space

Evaluation problem for FO in PSPACE

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$

Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
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Question:
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$\underbrace{2 \cdot \log(|G|) + \cdots + 2 \cdot \log(|G|)}_{\leq |\phi| \text{ times}} + k \cdot \log(|\alpha|+|G|)$ space

Combined, Query, and Data complexities

[Vardi, 1982]

Problem: Usual scenario in database

A **database** of size 10^6

A **query** of size 100

Input:

Combined, Query, and Data complexities

[Vardi, 1982]

Problem: Usual scenario in database

A **database** of size 10^6

A **query** of size 100

Input: • **query** +

Combined, Query, and Data comp

Problem: Usual scen

Input: • **query** +

database

Combined, Query, and Data compl

Problem: Usual scen

Input: • **query** +

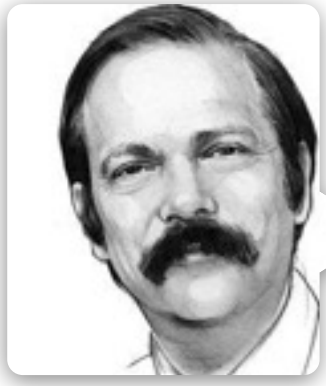
database

But we don't distinguish this in the analysis:

$$\mathrm{TIME}(2^{|\text{query}|} + |\text{data}|) = \mathrm{TIME}(|\text{query}| + 2^{|\text{data}|})$$

Combined, Query, and Data complexities

[Vardi, 1982]



Query and data play very **different roles.**

Separation of concerns: How the resources grow with respect to

- the size of the data
- the query size

Combined, Query, and Data complexities

Combined complexity: input size is |query| + |data|

Query complexity (|data| fixed): input size is |query|

Data complexity (|query| fixed): input size is |data|

Combined, Query, and Data complexities

Combined complexity: input size is |query| + |data|

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$O(2^{|query|} + |data|)$ is
- exponential in **combined** complexity
- exponential in **query** complexity
- linear in **data** complexity

$O(|query| + 2^{|data|})$ is
- exponential in **combined** complexity
- linear in **query** complexity
- exponential in **data** complexity

Question

What is the data, query and combined complexity
for the evaluation problem for FO?

Remember: **data** complexity, input size: |data|

query complexity, input size: |query|

combined complexity, input size: |data| + |query|

$|\phi| \cdot 2 \cdot \log(|G|) + k \cdot \log(|\alpha|+|G|)$ space

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What is the data, query and combined complexity
for the evaluation problem for FO?

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combined complexity, input size: |data| + |query|

$|\phi| \cdot 2 \cdot \log(|G|) + k \cdot \log(|\alpha| + |G|)$ space

O(log(|data|)·|query|) space

PSPACE combined and query complexity

LOGSPACE data complexity

Evaluation pb for FO is PSPACE-complete

(combined complexity)

PSPACE-complete problem: **QBF**

(satisfaction of Quantified Boolean Formulas)

QBF = a boolean formula with quantification over the truth values (T,F)

Evaluation pb for FO is PSPACE-complete

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PSPACE-complete problem: **QBF**

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$\exists p\ \forall q\ .\ (p \vee \neg q)$ where p,q range over {T,F}

Evaluation pb for FO is PSPACE-complete (combined complexity)

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Theorem: Evaluation for FO is PSPACE-complete (combined c.)

Evaluation pb for FO is PSPACE-complete

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Polynomial reduction **QBF** $\rightsquigarrow$ FO :

1. Given $\psi \in$ QBF,
let $\psi'(x)$ be the replacement of each 'p' with 'p=x' in ψ.

2. Note: $\exists x \, \psi'$ holds in a 2-element graph iff ψ is QBF-satisfiable

3. Test if $G \models_{\emptyset} \psi'$ for G=({v,v'},{})

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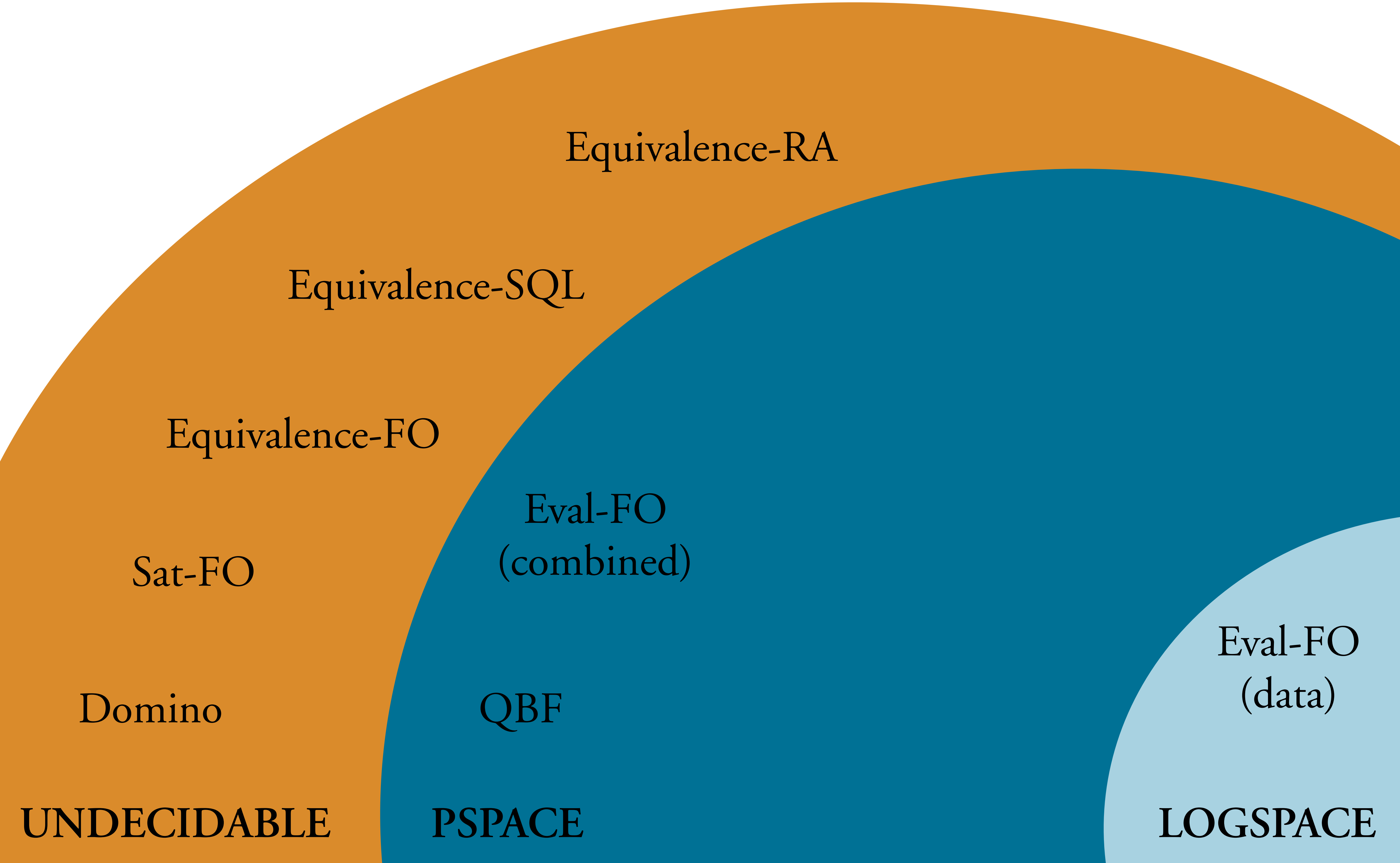
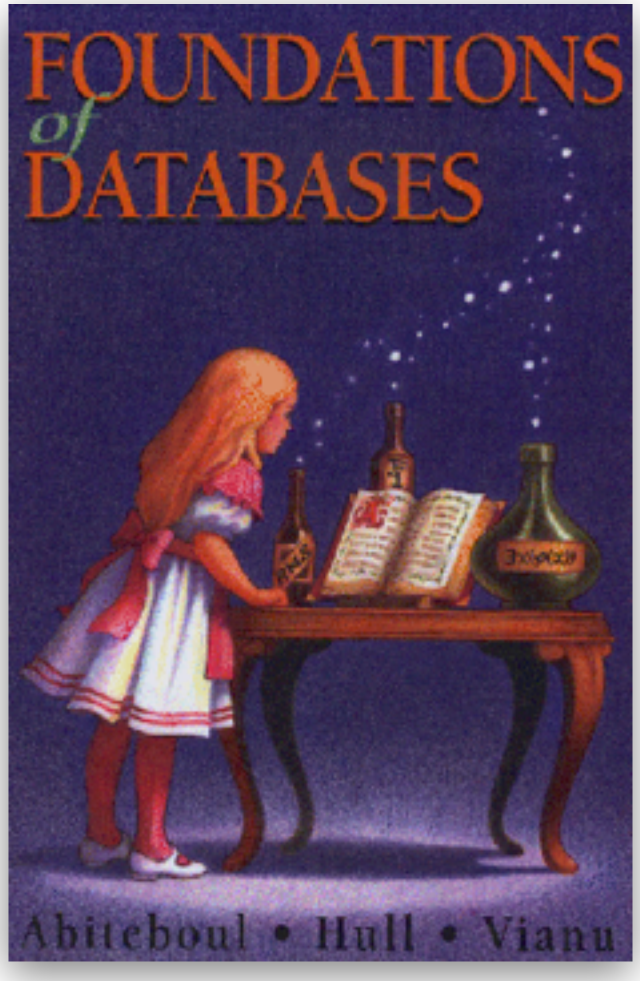
Recap


Equivalence-RA
Equivalence-SQL
Equivalence-FO
Sat-FO
Domino
UNDECIDABLE
Eval-FO
(combined)
QBF
PSPACE
Eval-FO
(data)
LOGSPACE

Bibliography

Abiteboul, Hull, Vianu, "Foundations of Databases", Addison-Wesley, 1995.

(freely available at http://webdam.inria.fr/Alice/)



Chapters 1, 2, 3